Chapter 5

Conclusions

This author expanded the reaction network of the Texas code to have the same 271 nuclear reactions and $\beta-$decays as the Thomas et al code. A comparison with the Thomas et al code revealed the correct way to linearize the equation for the time derivative of the isotope abundances $Y(i)$ so that the abundances the codes calculate would have second order convergence with decreasing timestep size. Also the comparison showed how to reduce the dependence of the codes' abundances/mass fractions and the agreement between the codes' results on computational factors such as the number of timesteps and the minimum value of $Y(i)$ permitted. For instance the mass fraction $X_{^4\mathrm{He}}$ agreed to within 10^{-4} by the end of the comparison.

By determining which reactions caused the largest positive or negative changes to $dY(i)/dt$ one can determine the temperature dependence of abundances $Y(i)$ and mass fractions X_i and follow the most important reactions to determine the nucleon flow of the Texas code. Isotopes such as ^{9}Be have destruction reactions that overwhelm the creation reactions, and so their mass fractions peak at the time of nucleosynthesis and then fall until they reach a final value once the destruction reaction rates have fallen below the expansion rate of the universe. Isotopes such ^{12}C have an overwhelming number of creation reactions instead, and their mass fractions asymptote to a final value. ^{9}Be, ^{10}B and ^{11}B have reactions that ultimately convert them to ^{4}He and ^{6}Li nuclei, ensuring that they have mass fractions much smaller than observational values [14]. The nuclear reactions sequentially create and destroy the isotopes ^{14}C, ^{15}N, ^{12}C, ^{13}N, ^{13}C, and ^{14}N again to form a cycle, though most nucleons in this cycle wind up in ^{12}C nuclei. Some leakage from ^{15}N goes on to form ^{16}O, though.

The codes' means of solving for $Y(i)$ provided a model for solving the Boltzmann Transport Equation for the neutrino distribution functions $f_{\nu_e}(p)$ and $f_{\nu_\mu}(p) = f_{\nu_\tau}(p)$ as discretized functions for 100 different values of momentum p. Electrons and positrons heat neutrinos primarily e^-e^+ annihilation into $\nu_e\bar{\nu}_e$ pairs, and then neutrino-only reactions distribute energy from higher momentum ν_e's to lower momentum ν_e's and ν_μ's and ν_τ's The higher the momentum the later $f_{\nu_e}(p)$ and $f_{\nu_\mu}(p)$ reach their final values, with the highest momentum ν_e's annihilating to form e^-e^+ pairs and decreasing $f_{\nu_e}(p)$. For $\eta_{10} = 3.0$ low momentum neutrinos whose $f_{\nu_i}(p)$'s reach their final values before $T = 2$ GK determine the increases of energy densities ρ_e and $\rho_\mu = \rho_\tau$, of 0.935 % and 0.384 % respectively. By increasing f_{ν_e} heating increases the conversion rate $n + \nu_e \rightarrow p + e^-$, and that effect alone decreases the amount of neutrons present to produce ^{4}He. But transfer of energy from the electrons and positrons to the neutrinos reduces the electromagnetic temperature T. That reduction leads to a clock effect where nucleosynthesis occurs at an earlier time. Ultimately nucleosynthesis occurs with more neutrons present then in the case without heating, but $X_{^4\mathrm{He}}$ increases only by an amount of from

2 to 1.1 $\times 10^{-4}$, with clock effect getting smaller with increasing η_{10}. $X_{^4\mathrm{He}}$ increases by 1.60 $\times 10^{-4}$ for $\eta_{10} = 3.0$ for instance.

From the Texas code this author constructed the Texas IBBN code, a code for a spherically symmetric baryon inhomogeneous model of the universe that starts at $T = 100$ GK with a high density inner region and a low density outer region. In models with $\eta_{10} = 3.0$ and distance scales r_i less than 25000 cm conversion reactions convert protons to neutrons in the high density shells and the neutrons diffuse to the low density shells where some neutrons become protons again. But fewer protons get redistributed as r_i increases and diffusion occurs later. At $r_i = 25000$ cm the inner shells retain their protons. Nucleosynthesis then occurs earlier in the inner shells and neutrons back diffuse into the inner shells where most of the ^{4}He is produced leading to an increasing $X_{^4\mathrm{He}}$. As r_i increases to 7.9 $\times 10^5$ cm neutrons can't back diffuse to all the inner shells, and $X_{^4\mathrm{He}}$ decreases instead. At $r_i = 7.9 \times 10^5$ cm diffusion occurs at the same time as nucleosynthesis, and $X_{^4\mathrm{He}}$ bottoms out. For higher r_i the high and low density shells behave independently, averaging out to a larger $X_{^4\mathrm{He}}$. $X_{^4\mathrm{He}}$ undergoes similar behavior for other values of η_{10}.

Neutrinos, electrons and positrons have homogeneous distributions in this IBBN model. Neutrino heating makes neutron diffusion occur earlier because heating reduces T at a given time. Heating has this effect throughout the ranges of η_{10} and $r(i)$ observed. So the increase in $X_{^4\mathrm{He}}$ due to neutrino heating remains in the same order of magnitude for those ranges, and the graph for $\Delta X_{^4\mathrm{He}}$. Neutrino heating also reduces $\log[Y(\mathrm{d})/Y(\mathrm{n})]$ by a small amount, on the order of 10^{-3}. This reduction turns out to be an order of magnitude less than the reduction $\log[Y(\mathrm{d})/Y(\mathrm{n})]$ would have by increasing η_{10} to increase $X_{^4\mathrm{He}}$ as much as the heating efffect does. Further, for high

$r(i)$ where the high and low density shells act independently heating increases $\log[Y(\mathrm{d})/Y(\mathrm{n})]$, possibly due to increased deuterium production in the low density shells. Neutrino heating might have an effect on ^{4}He and deuterium production distinctive from other phenomena. If observations can be measured accurate to the order of the effect neutrino heating could be distinguished in observations.

This dissertation looked at one particular baryon distribution, and a particular boundary radius $r_b = 0.5r_i$ and density contrast $R_\rho = 800$. This author would like to vary these parameters to see how the mass fractions with and without heating would change. This dissertation has already mentioned the electroweak phase transistion as a means of producing a model with a low density inner region and a high density outer region. This results from this inhomogeneous distribution should be compared to the dissertation's results, as well as the results from models with geometries other than spherical. The Texas IBBN code can be modified to accomodate all those alternative models.

Chapter 4 featured contour maps of ^{4}He and deuterium. The Texas IBBN code can produce maps for any of its heavier isotopes as well. It would be interesting to see how ^{9}Be, ^{10}B and ^{11}B, the isotopes in the six-isotope cycle, and ^{16}O change with different r_i and η_{10}. Experimental observations would put limits on the usefullness of the maps, though. Some observations include:

$$X_{^4\mathrm{He}} = 0.244 \pm 0.002$$

from HII region observations (Izotov & Thuan, 1998) [1],

$$Y(\mathrm{d})/Y(\mathrm{p}) \quad = \quad (3.4 \pm 0.3) \times 10^{-5}$$

from QSO observations in (Burles & Tytler, 1998) [50], and

$$\log[Y(^7\mathrm{L}i)/Y(\mathrm{d})] \quad = \quad -9.45 \pm 0.20$$

from the ^{7}Li abundance plateau (Pinsonneault et al, 1998) [51]. Putting 2σ constraints on Figures (4.2a) and (4.2b) leads to a range of η_{10} from 4.0 to 6.0 and a maximum $r(i)$ at 20000 cm. One should take these constraints into account when looking at contour maps for heavier isotopes and the heating effect. One can also anticipate that ^{4}He observations can improve by one order of magnitude so that $X_{^4He}$ can be measured accurate to the order of heating effect.

Appendix A

The Boltzmann Transport Equation

Let's start with C_{coll} as it appears in Equation (3.4), with p_i not rescaled.

$$
\begin{aligned}
C_{coll} \quad = \quad & \frac{1}{2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\int \frac{d^3p_3}{(2\pi)^3 2E_3}\int \frac{d^3p_4}{(2\pi)^3 2E_4} S|M|^2 \\
& \delta^4(p_1+p_2-p_3-p_4)(2\pi)^4\{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4(p_4) - \\
& f_1(p_1)f_2(p_2)[1-f_3(p_3)][1-f_4(p_4)]\}
\end{aligned}
$$

All the energies $E_i = \sqrt{m_i^2 + p_i^2}$.

A.1 d^3p_4

One can rewrite

$$\frac{d^3p_4}{2E_4} = d^4p_4\delta[(p_4)^2 - m_4^2]\Theta(p_4^0)$$

This switch makes the time component of p_4 to a variable to be integrated over. the δ function would pick out the two values of (p_4^0) such that the norm of p_4 equals m_4, and the step function picks out the positive value of p_4^0. We can now use $\delta^4(p_1 + p_2 - p_3 - p_4)$ to integrate over all four components of p_4

$$\begin{aligned} C_{coll} = & \frac{1}{8E_1(2\pi)^5}\int\frac{d^3p_2}{E_2}\int\frac{d^3p_3}{E_3}S|M|^2\Theta(E_1 + E_2 - E_3) \\ & \delta[(p_1)^2 + (p_2)^2 + (p_3)^2 + 2(p_1\cdot p_2 - p_1\cdot p_3 - p_2\cdot p_3) - m_4^2] \\ & \left\{[1 - f_1(p_1)][1 - f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(E_1 + E_2 - E_3)^2 - m_4^2}\right)\right. \\ & -f_1(p_1)f_2(p_2)[1 - f_3(p_3)] \\ & \left.\left[1 - f_4\left(\sqrt{(E_1 + E_2 - E_3)^2 - m_4^2}\right)\right]\right\} \end{aligned} \tag{A.1}$$

A.2 Spherical Coordinates

One can define a coordinate system such that p_1 lies along the z-axis and p_3 lies in the xz-plane. Then one can write d^3p_2 and d^3p_3 in terms of spherical coordinates

$$\begin{aligned} d^3p_2 &= p_2^2dp_2d(\cos\alpha)d\beta \\ d^3p_3 &= p_3^2dp_3d(\cos\theta)d\mu \end{aligned}$$

The dot products in the δ function in (A.1) can be written in terms of these coordinates. It turns out that nothing in the integrand depends on coordinate μ and so that coordinate can be integrated out. C_{coll} can then be written as.

$$
\begin{aligned}
C_{coll} &= \frac{1}{8E_1(2\pi)^4}\int dp_2 \frac{p_2^2}{E_2} d(\cos\alpha) d\beta \int dp_3 \frac{p_3^2}{E_3} d(\cos\theta) S|M|^2 \\
&\quad \Theta(E_1+E_2-E_3)\delta\{m_1^2+m_2^2+m_3^2-m_4^2+2(E_1E_2-E_1E_3-E_2E_3) \\
&\quad -2[p_1p_2\cos\alpha - p_1p_3\cos\theta - p_2p_3(\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\beta)]\} \\
&\quad \left\{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(E_1+E_2-E_3)^2-m_4^2}\right)\right. \\
&\quad \left. -f_1(p_1)f_2(p_2)[1-f_3(p_3)]\left[1-f_4\left(\sqrt{(E_1+E_2-E_3)^2-m_4^2}\right)\right]\right\} \qquad \text{(A.2)}
\end{aligned}
$$

A.3 Integration of β

One can use the remaining δ function to integrate over β. Set

$$
\begin{aligned}
Q &= m_1^2+m_2^2+m_3^2-m_4^2 \\
K &= Q+2(E_1E_2-E_1E_3-E_2E_3) \\
&\quad -2[p_1p_2\cos\alpha - p_1p_3\cos\theta - p_2p_3(\cos\alpha\cos\theta+\sin\alpha\sin\theta\cos\beta)]\}
\end{aligned}
$$

and find the values of β where $K=0$.

$$
\begin{aligned}
\cos\beta_i &= [2(E_2E_3-p_2p_3\cos\alpha\cos\theta) - Q + 2(E_1E_3-p_1p_3\cos\theta) \\
&\quad -2(E_1E_2-p_1p_2\cos\alpha)]/(2p_2p_3\sin\alpha\sin\theta) \\
\frac{dK}{d\beta} &= -2p_2p_3\sin\alpha\sin\theta\sin\beta \\
\frac{dK}{d\beta}|_{\beta_i} &= -\sqrt{H} \\
H &= (2p_2p_3\sin\alpha\sin\theta)^2 - [2(E_2E_3-p_2p_3\cos\alpha\cos\theta) \\
&\quad -Q+2(E_1E_3-p_1p_3\cos\theta)-2(E_1E_2-p_1p_2\cos\alpha)]^2
\end{aligned}
$$

The zero points depend on the other variables p_i, α, and θ. But values of those variables exist such that H will be negative. So the integrand must include a step function $\Theta(H)$ that will set the integral to zero in case H is not positive.

$$\begin{aligned}
C_{coll} &= \frac{1}{4E_1(2\pi)^4}\int dp_2\frac{p_2^2}{E_2}d(\cos\alpha)\int dp_3\frac{p_3^2}{E_3}d(\cos\theta)S|M|^2\Theta(E_1+E_2-E_3) \\
&\quad \frac{\Theta(H)}{\sqrt{H}}\left\{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(E_1+E_2-E_3)^2-m_4^2}\right)\right. \\
&\quad \left.-f_1(p_1)f_2(p_2)[1-f_3(p_3)]\left[1-f_4\left(\sqrt{(E_1+E_2-E_3)^2-m_4^2}\right)\right]\right\} \quad \text{(A.3)}
\end{aligned}$$

H can be rearranged as a quadratic function of $\cos\alpha$.

$$\begin{aligned}
H &= 4p_2^2[2p_1p_3\cos\theta-(p_1^2+p_3^2)]\cos^2\alpha \\
&\quad +4p_2(p_1-p_3\cos\theta)[2(E_1E_2-E_1E_3-E_2E_3)+Q+2p_1p_3\cos\theta]\cos\alpha \\
&\quad +4p_2^2p_3^2(1-\cos^2\theta)-\{Q+2[(E_1E_2-E_1E_3-E_2E_3)+p_1p_3\cos\theta]\}^2
\end{aligned}$$

A.4 Integration of α

$S|M|^2$ for each reaction can be written in terms of p_i, α, and θ. Sections (A.4) and (A.5) will look at reactions where particle # 1 is ν_e.

$\nu_{\mathrm{e}}+\nu_{\mathrm{e}}\to\nu_{\mathrm{e}}+\nu_{\mathrm{e}}$, $\nu_{\mathrm{e}}+\nu_i\to\nu_{\mathrm{e}}+\nu_i$

$$S|M|^2 = 32G_F^2p_1^2p_2^2(1-\cos\alpha)^2$$

$\nu_{\mathrm{e}}+\bar{\nu}_i\to\nu_{\mathrm{e}}+\bar{\nu}_i$, $\nu_{\mathrm{e}}+\bar{\nu}_{\mathrm{e}}\to\nu_i+\bar{\nu}_i$

$$S|M|^2 = 32G_F^2p_1^2\{p_2^2\cos^2\alpha-2p_2[p_2-p_3(1-\cos\theta)]\cos\alpha+$$

$$[p_2 - p_3(1-\cos\theta)]^2\}$$

$\nu_{\rm e} + \bar{\nu}_{\rm e} \to \nu_{\rm e} + \bar{\nu}_{\rm e}$

$$\begin{aligned} S|M|^2 &= 128G_F^2 p_1^2\{p_2^2\cos^2\alpha - 2p_2[p_2 - p_3(1-\cos\theta)]\cos\alpha + \\ &\quad [p_2 - p_3(1-\cos\theta)]^2\} \end{aligned}$$

$\nu_{\rm e}\, + {\rm e}^- \to \nu_{\rm e}\, + {\rm e}^-$

$$\begin{aligned} S|M|^2 &= 32G_F^2\langle (C_A + C_V)^2 p_1^2 (E_2 - p_2\cos\alpha)^2 \\ &\quad -(C_V^2 - C_A^2)m^2 p_1 p_3 (1-\cos\theta) \\ &\quad +(C_A - C_V)^2 p_1^2\{p_2^2\cos^2\alpha - 2p_2[E_2 - p_3(1-\cos\theta)]\cos\alpha + \\ &\quad [E_2 - p_3(1-\cos\theta)]^2\}\rangle \end{aligned}$$

$\nu_{\rm e}\, + {\rm e}^+ \to \nu_{\rm e}\, + {\rm e}^+$

$$\begin{aligned} S|M|^2 &= 32G_F^2\langle (C_A + C_V)^2 p_1^2\{p_2^2\cos^2\alpha \\ &\quad -2p_2[E_2 - p_3(1-\cos\theta)]\cos\alpha + [E_2 - p_3(1-\cos\theta)]^2\} \\ &\quad +(C_A - C_V)^2 p_1^2 (E_2 - p_2\cos\alpha)^2 - (C_V^2 - C_A^2)m^2 p_1 p_3(1-\cos\theta)\rangle \end{aligned}$$

$\nu_{\rm e} + \bar{\nu}_{\rm e} \to {\rm e}^-\, + {\rm e}^+$

$$\begin{aligned} S|M|^2 &= 32G_F^2\langle (C_A + C_V)^2 p_1^2\{p_2^2\cos^2\alpha \\ &\quad -2p_2[p_2 - (E_3 - p_3\cos\theta)]\cos\alpha + [p_2 - (E_3 - p_3\cos\theta)]^2\} \\ &\quad +(C_A - C_V)^2 p_1^2 (E_3 - p_3\cos\theta)^2 + (C_V^2 - C_A^2)m^2 p_1 p_2(1-\cos\alpha)\rangle \end{aligned}$$

All the matrix elements are quadratic functions in $\cos\alpha$. One can derive general equations for integrals involving ax^2+bx+c and plug them into each matrix element.

$$\begin{aligned}
\int_{-1}^{1} dx \frac{\Theta(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} &= \frac{\pi}{\sqrt{-a}}\Theta(b^2-4ac) \\
\int_{-1}^{1} dx \frac{x\Theta(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} &= -\frac{b}{2a}\frac{\pi}{\sqrt{-a}}\Theta(b^2-4ac) \\
\int_{-1}^{1} dx \frac{x^2\Theta(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} &= \frac{1}{2a}(\frac{3b^2}{4a}-c)\frac{\pi}{\sqrt{-a}}\Theta(b^2-4ac)
\end{aligned}$$

The $\cos^2\alpha$ factor in H will always be < 0. Therefore if H has any positive values it'll be positive between its roots.:

$$\begin{aligned}
\cos\alpha_+ &= \frac{-b-\sqrt{b^2-4ac}}{2a} \\
\cos\alpha_- &= \frac{-b+\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

The step function within the integral assures that only the region between the roots give a non-zero contribution to the integral. The integrals will equal zero if H is negative everywhere, and the roots will be imaginary. So the integral solutions include the step function $\Theta(b^2-4ac)$ to account for that occurence.

A.4.1 $\nu_e + \nu_e \to \nu_e + \nu_e$, $\nu_e + \nu_i \to \nu_e + \nu_i$

For instance for the reactions $\nu_e + \nu_e \to \nu_e + \nu_e$ and $\nu_e + \nu_i \to \nu_e + \nu_i$ the integral written out is:

$$C_{coll} = \frac{1}{4p_1(2\pi)^4}\int dp_2 p_2 d(\cos\alpha)\int dp_3 p_3 d(\cos\theta)\Theta(p_1+p_2-p_3)\frac{\Theta(H)}{\sqrt{H}}$$

$$
\begin{aligned}
& 32G_F^2p_1^2p_2^2(1-2\cos\alpha+\cos^2\alpha) \\
& \{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4(p_1+p_2-p_3)- \\
& f_1(p_1)f_2(p_2)[1-f_3(p_3)][1-f_4(p_1+p_2-p_3)]\} \\
H \quad = \quad & -4p_2^2[(p_1^2+p_3^2)-2p_1p_3\cos\theta]\cos^2\alpha \\
& +8p_2(p_1-p_3\cos\theta)(p_1p_2-p_2p_3-p_1p_3+p_1p_3\cos\theta)\cos\alpha \\
& +4\{p_2^2p_3^2(1-\cos^2\theta)-(p_1p_2-p_2p_3-p_1p_3+p_1p_3\cos\theta)^2\}
\end{aligned}
$$

Plugging the general equations into this integral yields:

$$
\begin{aligned}
C_{coll} \quad = \quad & \frac{G_F^2p_1}{4\pi^4}\int dp_2\int dp_3p_3d(\cos\theta)\Theta(p_1+p_2-p_3)\frac{\pi}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}} \\
& \Theta\{64p_1p_2^2p_3^3(1+\cos\theta)(1-\cos\theta)^2[p_1p_3(\cos\theta-1) \\
& +2p_2(p_1+p_2-p_3)]\} \\
& \Bigg[p_2^2+2p_2\left(\frac{-p_1p_3^2\cos^2\theta+p_3[p_1^2-(p_1p_2-p_2p_3-p_1p_3)]\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& \left.+\frac{p_1(p_1p_2-p_2p_3-p_1p_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)+\frac{3}{2}\left(\frac{-p_1p_3^2\cos^2\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& \left.+\frac{p_3[p_1^2-(p_1p_2-p_2p_3-p_1p_3)]\cos\theta+p_1(p_1p_2-p_2p_3-p_1p_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)^2 \\
& -\frac{1}{2}\left(\frac{-p_3^2(p_1^2+p_2^2)\cos^2\theta-2p_1p_3(p_1p_2-p_2p_3-p_1p_3)\cos\theta+p_2^2p_3^2}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& \left.-\frac{(p_1p_2-p_2p_3-p_1p_3)^2}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)\Bigg] \\
& \{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4(p_1+p_2-p_3) \\
& -f_1(p_1)f_2(p_2)[1-f_3(p_3)][1-f_4(p_1+p_2-p_3)]\} \qquad \text{(A.4)}
\end{aligned}
$$

In anticipation of the next section the terms in the integrand are arranged as polynomials of $\cos\theta$. All the other reactions between only neutrinos have expressions for C_{coll} similar to Equation (A.4)

The only difference between C_{coll} for $\nu_{\rm e} + \nu_{\rm e} \to \nu_{\rm e} + \nu_{\rm e}$ and C_{coll} for $\nu_{\rm e} + \nu_i \to \nu_{\rm e} + \nu_i$ is that in the phase space factor all the $f_i = f_{\nu_{\rm e}}$ for the first reaction while $f_{1,3} = f_{\nu_{\rm e}}$ and $f_{2,4} = f_{\nu_{\mu,\tau}}$ for the second reaction. In the subsections that list more than one reaction the difference in C_{coll} between the reactions listed is only in the phase space factor.

A.4.2 $\nu_e + \bar{\nu}_i \to \nu_e + \bar{\nu}_i$, $\nu_e + \bar{\nu}_e \to \nu_i + \bar{\nu}_i$

$$
\begin{aligned}
C_{coll} = {} & \frac{G_F^2 p_1}{4\pi^4} \int dp_2 \int dp_3 p_3 d(\cos\theta)\Theta(p_1 + p_2 - p_3) \\
& \Theta\{64 p_1 p_2^2 p_3^3 (1+\cos\theta)(1-\cos\theta)^2 \\
& [p_1 p_3(\cos\theta - 1) + 2p_2(p_1 + p_2 - p_3)]\} \\
& \frac{\pi}{\sqrt{p_1^2 + p_3^2 - 2p_1 p_3 \cos\theta}} \Big[p_3^2 \cos^2\theta + 2p_3(p_2 - p_3)\cos\theta + (p_2 - p_3)^2 \\
& +2\left(\frac{-p_1 p_3^3 \cos^3\theta + [p_1(p_1 - 2p_2 + 2p_3) + p_2 p_3] p_3^2 \cos^2\theta}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \right. \\
& + \frac{[(p_1 - p_2 + p_3)(p_1 p_2 - p_2 p_3 - p_1 p_3) + p_1^2(p_2 - p_3)] p_3 \cos\theta}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \\
& \left. + \frac{p_1(p_2 - p_3)(p_1 p_2 - p_2 p_3 - p_1 p_3)}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \right) + \frac{3}{2} \left(\frac{-p_1 p_3^2 \cos^2\theta}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \right. \\
& \left. + \frac{p_3[p_1^2 - (p_1 p_2 - p_2 p_3 - p_1 p_3)]\cos\theta + p_1(p_1 p_2 - p_2 p_3 - p_1 p_3)}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \right)^2 \\
& - \frac{1}{2} \left(\frac{-p_3^2(p_1^2 + p_2^2)\cos^2\theta - 2p_1 p_3(p_1 p_2 - p_2 p_3 - p_1 p_3)\cos\theta}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \right. \\
& \left. + \frac{p_2^2 p_3^2 - (p_1 p_2 - p_2 p_3 - p_1 p_3)^2}{2p_1 p_3 \cos\theta - (p_1^2 + p_3^2)} \right) \Big] \\
& \{[1 - f_1(p_1)][1 - f_2(p_2)] f_3(p_3) f_4(p_1 + p_2 - p_3) \\
& - f_1(p_1) f_2(p_2)[1 - f_3(p_3)][1 - f_4(p_1 + p_2 - p_3)]\}
\end{aligned} \tag{A.5}
$$

A.4.3 $\nu_e + \bar{\nu}_e \to \nu_e + \bar{\nu}_e$

Same as Equation (A.5), with an extra factor of 4.

A.4.4 $\nu_e + e^- \to \nu_e + e^-$

Keeping the $(C_A+C_V)^2$, $(C_A-C_V)^2$, and $(C_V^2-C_A^2)$ terms separate from each other makes the analytical work in this section easier to do.

$$
\begin{aligned}
C_{coll} \;=\;& \frac{G_F^2}{4\pi^4}\int dp_2\frac{p_2}{E_2}\int dp_3 p_3 d(\cos\theta)\Theta(p_1+E_2-p_3)\frac{\pi}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}} \\
& \Theta\Big\{64p_2^2p_3^2(\cos\theta+1)(\cos\theta-1)[p_1^2p_3^2\cos^2\theta+ \\
& 2p_1p_3(p_2^2+p_1E_2-p_1p_3-E_2p_3)\cos\theta-p_2^2(p_1^2+p_3^2) \\
& +(p_1E_2-p_1p_3-E_2p_3)^2]\Big\}\,\{(C_A+C_V)^2p_1\Big[E_2^2 \\
& +2E_2\left(\frac{-p_1p_3^2\cos^2\theta+p_3[p_1^2-(p_1E_2-E_2p_3-p_1p_3)]\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& \left.+\frac{p_1(p_1E_2-E_2p_3-p_1p_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)+\frac{3}{2}\left(\frac{-p_1p_3^2\cos^2\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& \left.+\frac{p_3[p_1^2-(p_1E_2-E_2p_3-p_1p_3)]\cos\theta+p_1(p_1E_2-E_2p_3-p_1p_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)^2 \\
& -\frac{1}{2}\left(\frac{-p_3^2(p_1^2+p_2^2)\cos^2\theta-2p_1p_3(p_1E_2-E_2p_3-p_1p_3)\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& \left.+\frac{p_2^2p_3^2-(p_1E_2-E_2p_3-p_1p_3)^2}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)\Big]-(C_V^2-C_A^2)m_e^2p_3(1-\cos\theta) \\
& +(C_A-C_V)^2p_1\Big[p_3^2\cos^2\theta+2p_3(E_2-p_3)\cos\theta+(E_2-p_3)^2 \\
& +2\left(\frac{-p_1p_3^3\cos^3\theta+[p_1(p_1-2E_2+2p_3)+E_2p_3]p_3^2\cos^2\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
& +\frac{[(p_1-E_2+p_3)(p_1E_2-E_2p_3-p_1p_3)+p_1^2(E_2-p_3)]p_3\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{p_1(E_2-p_3)(p_1E_2-E_2p_3-p_1p_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\bigg)+\frac{3}{2}\bigg(\frac{-p_1p_3^2\cos^2\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\\
&+\frac{p_3[p_1^2-(p_1E_2-E_2p_3-p_1p_3)]\cos\theta+p_1(p_1E_2-E_2p_3-p_1p_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\bigg)^2\\
&-\frac{1}{2}\bigg(\frac{-p_3^2(p_1^2+p_2^2)\cos^2\theta-2p_1p_3(p_1E_2-E_2p_3-p_1p_3)\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\\
&+\frac{p_2^2p_3^2-(p_1E_2-E_2p_3-p_1p_3)^2}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\bigg)\bigg]\bigg\}\\
&\bigg\{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(p_1+E_2-p_3)^2-m_e^2}\right)\\
&-f_1(p_1)f_2(p_2)[1-f_3(p_3)]\left[1-f_4\left(\sqrt{(p_1+E_2-p_3)^2-m_e^2}\right)\right]\bigg\} \quad \text{(A.6)}
\end{aligned}
$$

The $(C_A+C_V)^2$ term is the same as Equation (A.4) while the $(C_A-C_V)^2$ term is the same as Equation (A.5), except that p_2 is replaced with E_2.

$f_{2,4}=f_e$, the Fermi-Dirac distribution function of electrons.

A.4.5 $\nu_e + e^+ \to \nu_e + e^+$

Same as Equation (A.6), with $-C_V$ replacing C_V.

A.4.6 $\nu_e + \bar{\nu}_e \to e^- + e^+$

$$
\begin{aligned}
C_{coll} \;=\; & \frac{G_F^2}{4\pi^4}\int dp_2\int dp_3\frac{p_3^2}{E_3}d(\cos\theta)\Theta(p_1+E_2-p_3)\frac{\pi}{\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\\
&\Theta\{64p_2^2p_3^2(\cos\theta+1)(\cos\theta-1)[p_1^2p_3^2\cos^2\theta+\\
&2p_1p_3(p_1+p_2)(p_2-E_3)\cos\theta+(p_1+p_2)^2E_3^2-\\
&2p_1p_2(p_1+p_2)E_3-p_2^2p_3^2]\}\{(C_V^2-C_A^2)m_e^2\\
&\bigg[p_2+\bigg(\frac{-p_1p_3^2\cos^2\theta+p_3[p_1^2-(p_1p_2-p_2E_3-p_1E_3)]\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\\
&+\frac{p_1(p_1p_2-p_2E_3-p_1E_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\bigg)\bigg]+(C_A-C_V)^2p_1(E_3-p_3\cos\theta)^2
\end{aligned}
$$

$$
\begin{aligned}
&+(C_A+C_V)^2 p_1 \Big[p_3^2\cos^2\theta + 2p_3(p_2-E_3)\cos\theta + (p_2-E_3)^2 \\
&+2\left(\frac{-p_1p_3^3\cos^3\theta + [p_1(p_1-2p_2+2E_3)+p_2E_3]p_3^2\cos^2\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
&+\frac{[(p_1-p_2+E_3)(p_1p_2-p_2E_3-p_1E_3)+p_1^2(p_2-E_3)]p_3\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)} \\
&\left.+\frac{p_1(p_2-E_3)(p_1p_2-p_2E_3-p_1E_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right) + \frac{3}{2}\left(\frac{-p_1p_3^2\cos^2\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
&\left.+\frac{p_3[p_1^2-(p_1p_2-p_2E_3-p_1E_3)]\cos\theta + p_1(p_1p_2-p_2E_3-p_1E_3)}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)^2 \\
&-\frac{1}{2}\left(\frac{-p_3^2(p_1^2+p_2^2)\cos^2\theta - 2p_1p_3(p_1p_2-p_2E_3-p_1E_3)\cos\theta}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right. \\
&\left.+\frac{p_2^2p_3^2-(p_1p_2-p_2E_3-p_1E_3)^2}{2p_1p_3\cos\theta-(p_1^2+p_3^2)}\right)\Big]\Big\} \\
&\Big\{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(p_1+p_2-E_3)^2-m_e^2}\right) \\
&-f_1(p_1)f_2(p_2)[1-f_3(p_3)]\left[1-f_4\left(\sqrt{(p_1+p_2-E_3)^2-m_e^2}\right)\right]\Big\} \qquad \text{(A.7)}
\end{aligned}
$$

The $(C_A+C_V)^2$ term is the same as Equation (A.5) except that in the numerators the variable p_3 is replaced with E_3. The $(C_V^2-C_A^2)$ term has similarities with Equation (A.4).

A.5 Integration of θ

Once again, the forms of all the integrals can be fit into a small number of general equations listed below.

$$
\begin{aligned}
I_1 &= \int dx \frac{d_1x^3+d_2x^2+d_3x+d_4}{(bx-a)\sqrt{a-bx}} \\
I_1 &= -\frac{2d_1}{b}\left(\frac{x^3}{\sqrt{a-bx}} + \frac{6}{b}x^2\sqrt{a-bx} + \frac{8}{b^2}x(a-bx)^{3/2} + \frac{16}{5b^3}(a-bx)^{5/2}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{2d_2}{b}\left(\frac{x^2}{\sqrt{a-bx}}+\frac{4}{b}x\sqrt{a-bx}+\frac{8}{3b^2}(a-bx)^{3/2}\right) \\
& -\frac{2d_3}{b}\left(\frac{x}{\sqrt{a-bx}}+\frac{2}{b}\sqrt{a-bx}\right)-\frac{2d_4}{b\sqrt{a-bx}} \\
I_2 &= \int dx\frac{d_1x^2+d_2x+d_3}{\sqrt{a-bx}} \\
I_2 &= -\frac{2d_1}{b}\left(x^2\sqrt{a-bx}+\frac{4}{3b}x(a-bx)^{3/2}+\frac{8}{15b^2}(a-bx)^{5/2}\right) \\
& -\frac{2d_2}{b}\left(x\sqrt{a-bx}+\frac{2}{3b}(a-bx)^{3/2}\right)-\frac{2d_3}{b}\sqrt{a-bx} \\
I_3 &= \int dx\left(\frac{d_1x^2+d_2x+d_3}{bx-a}\right)^2\frac{1}{\sqrt{a-bx}} \\
I_3 &= \frac{2d_1^2}{b}\left(\frac{x^4}{3(a-bx)^{3/2}}-\frac{8}{3b}\frac{x^3}{\sqrt{a-bx}}-\frac{16}{b^2}x^2\sqrt{a-bx}-\frac{64}{3b^3}x(a-bx)^{3/2}\right. \\
& \left.-\frac{128}{15b^4}(a-bx)^{5/2}\right) \\
& +\frac{4d_1d_2}{b}\left(\frac{x^3}{3(a-bx)^{3/2}}-\frac{2}{b}\frac{x^2}{\sqrt{a-bx}}-\frac{8}{b^2}x\sqrt{a-bx}-\frac{16}{3b^3}(a-bx)^{3/2}\right) \\
& +\frac{2(d_2^2+2d_1d_3)}{3b}\left(\frac{x^2}{(a-bx)^{3/2}}-\frac{4}{b}\frac{x}{\sqrt{a-bx}}-\frac{8}{b^2}\sqrt{a-bx}\right) \\
& +\frac{4d_2d_3}{3b}\left(\frac{x}{(a-bx)^{3/2}}+\frac{2}{b\sqrt{a-bx}}\right)-\frac{2d_3^2}{3b(a-bx)^{3/2}}
\end{aligned}
$$

The step functions in Equations (A.4-7) determine the upper and lower limits θ_+, θ_- that θ gets integrated over. This author was concerned that there could exist combinations of θ_+, θ_-, p_1, and p_3 that would set the denominators in the integral solutions to zero. But it turns out that the general equations above are such that in every instance one can cancel out the denominators at θ_+ and θ_-.

A.5.1 $\nu_e + \nu_e \to \nu_e + \nu_e$, $\nu_e + \nu_i \to \nu_e + \nu_i$

In Equation (A.4) $\cos\theta_+ = 1$ for all values of p_1, p_2, and p_3. But the value of $\cos\theta_-$ is conditional

$$\begin{aligned}
\cos\theta_- &= -1, p_3 < p_2 \\
\cos\theta_- &= 1 - \frac{2p_2(p_1+p_2-p_3)}{p_1 p_3}, p_3 > p_2
\end{aligned}$$

Use the general equations listed above to integrate Equation (A.4) between θ_- and θ_+, and then reorganize the integrand into a polynomial of $\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}$.

$$\begin{aligned}
C_{coll} &= \frac{G_F^2}{4\pi^3 p_1^2}\int_0^\infty dp_2 \int_0^{p_1+p_2} dp_3 \bigg| -\frac{4}{5}\left(p_1^2+p_3^2-2p_1p_3\cos\theta\right)^{5/2} \\
&+\frac{1}{3}\{2p_1p_2-(p_1-p_2)^2-6\left[p_1p_3\cos\theta+(p_1p_2-p_2p_3-p_1p_3)\right. \\
&\left.-p_1(p_1-p_3\cos\theta)\right]\}(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2} \\
&+\{p_1[p_1p_3\cos\theta+(p_1p_2-p_2p_3-p_1p_3)][2(p_1-p_3\cos\theta)-p_1] \\
&-p_1p_2^2p_3\cos\theta-[p_1(p_1-p_3\cos\theta) \\
&-(p_1p_3\cos\theta-p_2p_3-p_1p_3)]^2\}\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta} \\
&+I_{den}|_{\cos\theta_-}^{1}\ \{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4(p_1+p_2-p_3) \\
&-f_1(p_1)f_2(p_2)[1-f_3(p_3)][1-f_4(p_1+p_2-p_3)]\} \qquad \text{(A.8)} \\
I_{den} &= \frac{(p_1p_2p_3)^2(1-\cos^2\theta)}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}} + p_1[p_1p_3\cos\theta+(p_1p_2-p_2p_3-p_1p_3)] \\
&\left(\frac{[2(p_1-p_3\cos\theta)-p_1][p_1p_3\cos\theta+(p_1p_2-p_2p_3-p_1p_3)]}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right. \\
&\left.-\frac{2p_1[(p_1-p_3\cos\theta+p_2)^2-p_2^2]}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right)
\end{aligned}$$

$$+\frac{p_1^2(p_1-p_3\cos\theta)^2[p_1p_3\cos\theta+(p_1p_2-p_2p_3-p_1p_3)]^2}{2(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2}}$$

I_{den} is the part of the integrand that has $\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}$ in the denominator. But when θ_+ or either of the two values of θ_- is plugged into I_{den} the denominators cancel out.

$$\begin{aligned}
\cos\theta_+ &= 1 \\
I_{den} &= -p_1p_2[p_1(p_1+p_2-p_3)+p_2p_3]|p_1-p_3| \\
\cos\theta_- &= -1 \\
I_{den} &= -p_1[p_1(p_2-p_3)-p_3(p_1+p_2)] \\
&\quad [(p_1+p_3)(p_1+p_2+p_3)+p_3(p_1-p_3)] \\
\cos\theta_- &= 1-\frac{2p_2(p_1+p_2-p_3)}{p_1p_3} \\
I_{den} &= p_2[p_1(p_1-p_3)+2p_2(p_1+p_2-p_3)] \\
&\quad [(p_1+p_2)(p_1+2p_2-p_3)+2p_2(p_1+p_2-p_3)]
\end{aligned}$$

The Texas code uses these expressions for I_{den}, using an IF-THEN loop to pick the correct expression.

p_1+p_2 is the total energy of the incoming neutrinos. So p_3 is integrated only from 0 to p_1+p_2. But p_2 can have any amount of energy. So one integrates p_2 up to infinity.

A.5.2 $\nu_e+\bar{\nu}_i\rightarrow\nu_e+\bar{\nu}_i$, $\nu_e+\bar{\nu}_e\rightarrow\nu_i+\bar{\nu}_i$

$$\begin{aligned}
C_{coll} &= \frac{G_F^2}{4\pi^3p_1^2}\int_0^\infty dp_2\int_0^{p_1+p_2}dp_3\left|-\frac{2}{15}(p_1^2+p_3^2-2p_1p_3\cos\theta)^{5/2}\right. \\
&\quad +\frac{1}{3}\{2[p_1(p_1-p_3\cos\theta)+p_2p_3]-(p_1^2+p_2^2)\}(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2}
\end{aligned}$$

$$
\begin{aligned}
& -\{p_1^2(p_1p_3\cos\theta + p_1p_2 - p_2p_3 - p_1p_3) + p_1p_2^2p_3\cos\theta \\
& +[p_1(p_1 - p_3\cos\theta) + p_2p_3]^2\}\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta} + \\
& I_{den}\Big|_{\cos\theta_-}^{1} \{[1 - f_1(p_1)][1 - f_2(p_2)]f_3(p_3)f_4(p_1 + p_2 - p_3) \\
& -f_1(p_1)f_2(p_2)[1 - f_3(p_3)][1 - f_4(p_1 + p_2 - p_3)]\} \qquad \text{(A.9)}
\end{aligned}
$$

$$
\begin{aligned}
I_{den} = & \left(\frac{(p_1p_2p_3)^2(1 - \cos^2\theta) - p_1^2(p_1p_3\cos\theta + p_1p_2 - p_2p_3 - p_1p_3)^2}{2\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} \right. \\
& -2p_1(p_1 - p_3\cos\theta) \times \\
& \left. \frac{(p_1p_3\cos\theta + p_1p_2 - p_2p_3 - p_1p_3)[p_1(p_1 + p_2 - p_3) + p_2p_3]}{2\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} \right) \\
& + \frac{p_1^2(p_1 - p_3\cos\theta)^2(p_1p_3\cos\theta + p_1p_2 - p_2p_3 - p_1p_3)^2}{2(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{3/2}}
\end{aligned}
$$

The values of θ_+ and θ_- for these reactions are the same as for the reactions of Equation (A.8).

$$
\begin{aligned}
\cos\theta_+ &= 1 \\
I_{den} &= -p_1p_2[p_1(p_1 + p_2 - p_3) + p_2p_3]|p_1 - p_3| \\
\cos\theta_- &= -1 \\
I_{den} &= -p_1[p_1(p_2 - p_3) - p_3(p_1 + p_2)][p_1(p_1 + p_2 - p_3) + p_2p_3] \\
\cos\theta_- &= 1 - \frac{2p_2(p_1 + p_2 - p_3)}{p_1p_3} \\
I_{den} &= p_2[p_1(p_1 - p_3) + 2p_2(p_1 + p_2 - p_3)][p_1(p_1 + p_2 - p_3) + p_2p_3]
\end{aligned}
$$

Equation (A.9) likewise has the same upper and lower limits for p_2 and p_3.

A.5.3 $\nu_e + \bar{\nu}_e \rightarrow \nu_e + \bar{\nu}_e$

Same as Equation (A.9), with an extra factor of 4.

A.5.4 $\nu_e + e^- \rightarrow \nu_e + e^-$

In this case the limits of θ can have the following values.

$$\begin{aligned}
\cos\theta_+ &= \min\left(1, 1 - \frac{E_2(p_1+E_2-p_3) - m_e^2 - p_2\sqrt{(p_1+E_2-p_3)^2 - m_e^2}}{p_1p_3}\right) \\
\cos\theta_- &= \max \\
&\left(-1, 1 - \frac{E_2(p_1+E_2-p_3) - m_e^2 + p_2\sqrt{(p_1+E_2-p_3)^2 - m_e^2}}{p_1p_3}\right)
\end{aligned}$$

The code determines which values are the correct ones.

The upper limit of p_3 changes from $p_1 + p_2$ to $p_1 + E_2 - m_e$ to account for the non-zero electron masses. One can combine the results of (A.8) and (A.9) to get:

$$\begin{aligned}
C_{coll} &= \frac{G_F^2}{4\pi^3 p_1^2}\int_0^\infty dp_2 \frac{p_2}{E_2}\int_0^{p_1+E_2-m_e} dp_3 \Big|(C_A + C_V)^2 \\
&\Big[-\frac{4}{5}(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{5/2} + \frac{1}{3}\{4p_1E_2 - (p_1^2 + p_2^2) \\
&-6\left[(p_1p_3\cos\theta + p_1E_2 - E_2p_3 - p_1p_3)\right. \\
&\left.-p_1(p_1 - p_3\cos\theta)\right]\}(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{3/2} \\
&+\{p_1(p_1p_3\cos\theta + p_1E_2 - E_2p_3 - p_1p_3)[2(p_1 - p_3\cos\theta) - p_1] \\
&-p_1p_2^2p_3\cos\theta - [p_1(p_1 - p_3\cos\theta) - (p_1p_3\cos\theta - E_2p_3 - p_1p_3)]^2\} \\
&\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}\Big] \\
&-(C_V^2 - C_A^2)m_e^2\left[\frac{1}{3}(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{3/2}\right. \\
&\left.-p_1p_3(1-\cos\theta)\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}\right] + (C_A - C_V)^2 \\
&\Big[-\frac{2}{15}(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{5/2} + \frac{1}{3}\{2[p_1(p_1 - p_3\cos\theta) + E_2p_3]
\end{aligned}$$

$$
\begin{aligned}
& -(p_1^2+p_2^2)\}(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2} \\
& -\{p_1^2(p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)+p_1p_2^2p_3\cos\theta \\
& +[p_1(p_1-p_3\cos\theta)+E_2p_3]^2\}\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}\Big] \\
& + I_{den}\Big|_{\cos\theta_-}^{\cos\theta_+} \\
& \Big\{[1-f_1(p_1)][1-f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(p_1+E_2-p_3)^2-m_{\mathrm{e}}^2}\right) \\
& -f_1(p_1)f_2(p_2)[1-f_3(p_3)]\left[1-f_4\left(\sqrt{(p_1+E_2-p_3)^2-m_{\mathrm{e}}^2}\right)\right]\Big\} \qquad \text{(A.10)}
\end{aligned}
$$

$$
\begin{aligned}
I_{den} \;=\; & (C_A+C_V)^2\left[\frac{(p_1p_2p_3)^2(1-\cos^2\theta)}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right. \\
& +p_1(p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)\times \\
& \left(\frac{[2(p_1-p_3\cos\theta)-p_1](p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right. \\
& \left.-\frac{2p_1[(p_1-p_3\cos\theta+E_2)^2-E_2^2]}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right) \\
& \left.+\frac{p_1^2(p_1-p_3\cos\theta)^2(p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)^2}{2(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2}}\right] \\
& +(C_A-C_V)^2 \\
& \left[\left(\frac{(p_1p_2p_3)^2(1-\cos^2\theta)-p_1^2(p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)^2}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right.\right. \\
& -2p_1(p_1-p_3\cos\theta)(p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)\times \\
& \left.\frac{[p_1(p_1+E_2-p_3)+E_2p_3]}{2\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}}\right) \\
& \left.+\frac{p_1^2(p_1-p_3\cos\theta)^2(p_1p_3\cos\theta+p_1E_2-E_2p_3-p_1p_3)^2}{2(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2}}\right]
\end{aligned}
$$

For all four values of θ_+ and θ_- the denominators in I_{den} disappear.

$$
\cos\theta_+ \;=\; 1
$$

$$
\begin{aligned}
I_{den} &= -(C_A + C_V)^2 p_1 E_2 [p_1(p_1 + E_2 - p_3) + E_2 p_3] |p_1 - p_3| \\
&\quad -(C_A - C_V)^2 p_1 E_2 [p_1(p_1 + E_2 - p_3) + E_2 p_3] |p_1 - p_3| \\
\cos\theta_- &= -1 \\
I_{den} &= -(C_A + C_V)^2 p_1 [p_1(E_2 - p_3) - p_3(p_1 + E_2)] \\
&\quad [(p_1 + p_3)(p_1 + E_2 + p_3) + p_3(p_1 - p_3)] \\
&\quad -(C_A - C_V)^2 p_1 [p_1(E_2 - p_3) - p_3(p_1 + E_2)] \\
&\quad [p_1(p_1 + E_2 - p_3) + E_2 p_3] \\
\cos\theta_+ &= 1 - \frac{E_2(p_1 + E_2 - p_3) - m_e^2 - p_2\sqrt{(p_1 + E_2 - p_3)^2 - m_e^2}}{p_1 p_3} \\
I_{den} &= (C_A + C_V)^2 L p_2 \\
&\quad [(p_1 + E_2)(p_1 - p_3) + p_2(p_2 - \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2})] \\
&\quad [(p_1 + E_2)(p_1 - p_3) + 2p_1 E_2 + 2p_2(p_2 - \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2})] \\
&\quad +(C_A - C_V)^2 L p_2 [p_1(p_1 + E_2 - p_3) + E_2 p_3] \\
&\quad [(p_1 + E_2)(p_1 - p_3) + p_2(p_2 - \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2})] \\
L &= \text{The Sign of } \left(p_2 - \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2}\right) \\
\cos\theta_- &= 1 - \frac{E_2(p_1 + E_2 - p_3) - m_e^2 + p_2\sqrt{(p_1 + E_2 - p_3)^2 - m_e^2}}{p_1 p_3} \\
I_{den} &= (C_A + C_V)^2 p_2 \\
&\quad [(p_1 + E_2)(p_1 - p_3) + p_2(p_2 + \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2})] \\
&\quad [(p_1 + E_2)(p_1 - p_3) + 2p_1 E_2 + 2p_2(p_2 + \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2})] \\
&\quad +(C_A - C_V)^2 p_2 [p_1(p_1 + E_2 - p_3) + E_2 p_3] \\
&\quad [(p_1 + E_2)(p_1 - p_3) + p_2(p_2 + \sqrt{(p_1 + E_2 - p_3)^2 - m_e^2})]
\end{aligned}
$$

A.5.5 $\nu_e + e^+ \rightarrow \nu_e + e^+$

Same as (A.10), with C_V replaced with $-C_V$.

A.5.6 $\nu_e + \bar{\nu}_e \rightarrow e^- + e^+$

Finally, this reaction has the following limits.:

$$\begin{aligned}
\cos\theta_+ &= \min\left(1, \frac{-(p_1+p_2)(p_2-E_3)+p_2\sqrt{(p_1+p_2-E_3)^2-m_e^2}}{p_1p_3}\right) \\
\cos\theta_- &= \max\left(-1, \frac{-(p_1+p_2)(p_2-E_3)-p_2\sqrt{(p_1+p_2-E_3)^2-m_e^2}}{p_1p_3}\right)
\end{aligned}$$

The combined energy $p_1 + p_2$ of the incoming neutrinos has to be enough to produce at least an electron/positron pair at rest. This is guaranteed by setting a lower limit p_{2-} to p_2

$$p_{2-} = \max(0, 2m_e - p_1)$$

p_3 then has an upper limit $\sqrt{(p_1+p_2-m_e)^2-m_e^2}$ to account for the energy needed to make the rest masses. C_{coll} is then:

$$\begin{aligned}
C_{coll} = &\frac{G_F^2}{4\pi^3 p_1^2}\int_{2m_e-p_1}^{\infty} dp_2 \int_0^{\sqrt{(p_1+p_2-m_e)^2-m_e^2}} dp_3 \frac{p_3}{E_3}\Big|(C_V^2 - C_A^2)m_e^2 \\
&\left[\frac{2}{3}(p_1^2+p_3^2-2p_1p_3\cos\theta)^{3/2} + [(p_1p_3\cos\theta - p_2E_3 - p_1E_3)\right. \\
&\left. -p_1(p_1-p_3\cos\theta)]\sqrt{p_1^2+p_3^2-2p_1p_3\cos\theta}\right] \\
&-(C_A - C_V)^2\left[\frac{2}{15}(p_1^2+p_3^2-2p_1p_3\cos\theta)^{5/2}\right.
\end{aligned}$$

$$
\begin{aligned}
& -\frac{2}{3}p_1(E_3 - p_3\cos\theta)(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{3/2} \\
& +p_1^2(E_3 - p_3\cos\theta)^2\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}\Big] + (C_A + C_V)^2 \\
& \Big[-\frac{2}{15}(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{5/2} + \frac{1}{3}\{2[p_1(p_1 - p_3\cos\theta) + p_2E_3] \\
& -(p_1^2 + p_2^2)\}(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{3/2} \\
& -\{p_1^2(p_1p_3\cos\theta + p_1p_2 - p_2E_3 - p_1E_3) + p_1p_2^2p_3\cos\theta \\
& +[p_1(p_1 - p_3\cos\theta) + p_2E_3]^2\}\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}\Big] + I_{den}\Big|_{\cos\theta_-}^{\cos\theta_+} \\
& \Big\{[1 - f_1(p_1)][1 - f_2(p_2)]f_3(p_3)f_4\left(\sqrt{(p_1 + p_2 - E_3)^2 - m_e^2}\right) \\
& -f_1(p_1)f_2(p_2)[1 - f_3(p_3)]\left[1 - f_4\left(\sqrt{(p_1 + p_2 - E_3)^2 - m_e^2}\right)\right]\Big\} \qquad (A.11)
\end{aligned}
$$

$$
\begin{aligned}
I_{den} \;=\; & (C_V^2 - C_A^2)m_e^2 \\
& \left[-\frac{p_1(p_1 - p_3\cos\theta)(p_1p_3\cos\theta + p_1p_2 - p_2E_3 - p_1E_3)}{\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}}\right] \\
& +(C_A + C_V)^2\Bigg[\Bigg(\frac{(p_1p_2p_3)^2(1 - \cos^2\theta)}{2\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} \\
& -\frac{p_1^2(p_1p_3\cos\theta + p_1p_2 - p_2E_3 - p_1E_3)^2}{2\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}} - 2p_1(p_1 - p_3\cos\theta)\times \\
& \frac{(p_1p_3\cos\theta + p_1p_2 - p_2E_3 - p_1E_3)[p_1(p_1 + p_2 - E_3) + p_2E_3]}{2\sqrt{p_1^2 + p_3^2 - 2p_1p_3\cos\theta}}\Bigg) \\
& +\frac{p_1^2(p_1 - p_3\cos\theta)^2(p_1p_3\cos\theta + p_1p_2 - p_2E_3 - p_1E_3)^2}{2(p_1^2 + p_3^2 - 2p_1p_3\cos\theta)^{3/2}}\Bigg]
\end{aligned}
$$

Plugging in the values of θ_+ and θ_- into I_{den}

$$
\begin{aligned}
\cos\theta_+ &= 1 \\
I_{den} &= -(C_V^2 - C_A^2)Lm_e^2p_1[p_2(p_1 - E_3) - p_1(E_3 - p_3)] \\
&\quad -(C_A + C_V)^2Lp_1[p_1(p_1 + p_2 - E_3) + p_2E_3]
\end{aligned}
$$

$$
\begin{aligned}
& [p_2(p_1 - E_3) - p_1(E_3 - p_3)] \\
L &= \text{The Sign of } (p_1 - p_3) \\
\cos\theta_- &= -1 \\
I_{den} &= -(C_V^2 - C_A^2)m_e^2 p_1[p_1(p_2 - p_3) - E_3(p_1 + p_2)] \\
& -(C_A + C_V)^2 p_1[p_1(p_1 + p_2 - E_3) + p_2 E_3] \\
& [p_1(p_2 - p_3) - E_3(p_1 + p_2)] \\
\cos\theta_+ &= \frac{-(p_1 + p_2)(p_2 - E_3) + p_2\sqrt{(p_1 + p_2 - E_3)^2 - m_e^2}}{p_1 p_3} \\
I_{den} &= (C_V^2 - C_A^2)Lm_e^2 p_2[(p_1 - E_3)(p_1 + p_2) \\
& +p_2(p_2 - \sqrt{(p_1 + p_2 - E_3)^2 - m_e^2})] \\
& +(C_A + C_V)^2 L p_2[p_1(p_1 + p_2 - E_3) + p_2 E_3] \\
& [(p_1 - E_3)(p_1 + p_2) + p_2(p_2 - \sqrt{(p_1 + p_2 - E_3)^2 - m_e^2})] \\
L &= \text{The Sign of } (p_2 - \sqrt{(p_1 + p_2 - E_3)^2 - m_e^2}) \\
\cos\theta_- &= \frac{-(p_1 + p_2)(p_2 - E_3) - p_2\sqrt{(p_1 + p_2 - E_3)^2 - m_e^2}}{p_1 p_3} \\
I_{den} &= (C_V^2 - C_A^2)m_e^2 p_2[(p_1 - E_3)(p_1 + p_2) \\
& +p_2(p_2 + \sqrt{(p_1 + p_2 - E_3)^2 - m_e^2})] \\
& +(C_A + C_V)^2 p_2[p_1(p_1 + p_2 - E_3) + p_2 E_3] \\
& [(p_1 - E_3)(p_1 + p_2) + p_2(p_2 + \sqrt{(p_1 + p_2 - E_3)^2 - m_e^2})]
\end{aligned}
$$

Equations (A.8) to (A.11) are added together in such a way as to account for all the reactions with ν_e as an incoming particle, and the resulting sum is the time derivative of $f_{\nu_e}(p_1)$.

A.6 Boltzmann Equation for μ and τ neutrinos

For the reactions with ν_i (ν_μ or ν_τ) as particle # 1 the phase space factors in the Boltzmann equation will of course be different from the preceding equations. But the matrix elements are the same as those listed in Section (A.4) (Almost. See below.) So one can simply match one of the equations between Equations (A.8) and (A.11) with each of these new reactions.

A.6.1 $\nu_i + \nu_i \to \nu_i + \nu_i$, $\nu_i + \nu_\mathrm{e} \to \nu_i + \nu_\mathrm{e}$, $\nu_i + \nu_j \to \nu_i + \nu_j$

C_{coll} equals Equation (A.8), with the phase space factor changed accordingly.

A.6.2 $\nu_i + \bar{\nu}_\mathrm{e} \to \nu_i + \bar{\nu}_\mathrm{e}$, $\nu_i + \bar{\nu}_j \to \nu_i + \bar{\nu}_j$, $\nu_i + \bar{\nu}_i \to \nu_\mathrm{e} + \bar{\nu}_\mathrm{e}$, $\nu_i + \bar{\nu}_i \to \nu_j + \bar{\nu}_j$

C_{coll} equals Equation (A.9) in these cases.

A.6.3 $\nu_i + \bar{\nu}_i \to \nu_i + \bar{\nu}_i$

Same as Equation (A.9) with an extra factor of 4.

A.6.4 $\nu_i + \mathrm{e}^- \to \nu_i + \mathrm{e}^-$

C_{coll} is the same as Equation (A.10), but because this is scattering between an electron and a μ, τ -neutrino C_A and C_V are replaced everywhere with $C_V - 1$ and $C_A - 1$.

A.6.5 $\nu_i + \mathbf{e}^+ \rightarrow \nu_i + \mathbf{e}^+$

Same as Equation (A.10), but C_A is replaced with $C_A - 1$ and C_V is replaced with $-(C_V - 1)$.

A.6.6 $\nu_i + \bar{\nu}_i \rightarrow \mathbf{e}^- + \mathbf{e}^+$

C_{coll} is the same as Equation (A.11), again with C_A and C_V replaced everywhere with $C_V - 1$ and $C_A - 1$.

A.7 The Rescaling of p_i

The p_i in Equations (A.8-11) are the physical momenta. There these expressions for C_{coll} equal the physical p_1 form

$$C_L = \frac{\partial}{\partial t} f_1(p_1, t) - \frac{\partial R}{\partial t} \frac{p_1}{R} \frac{\partial}{\partial p_1} f_1(p_1, t)$$

The second term accounts for the expansion of the universe. One should replace p_i with a corresponding $\tilde{p}_i$ such that the second term disappears in C_L.

$$\tilde{p}_i = \frac{p_i R}{k T_{\nu_o}}$$

This expression will suffice. The R in $(p_i R)/(kT_{\nu_o})$ alone accounts for the second term in C_L. R has to equal 1 when the neutrino temperature $T_\nu = T_{\nu_o} = 1.946\ ^o$K, the neutrino temperature for right now. In both the Texas code and the Thomas et al code R is initiallized to $R = T/T_{\nu_o}$ when $T = 100$ GK. When $\tilde{p}_i$ is plugged into the Fermi-Dirac distribution function

$$
\begin{aligned}
f_\nu(p_i,t) &= \frac{1}{e^{p_i/kT_\nu}+1} \\
f_\nu(\tilde{p}_i,t) &= \frac{1}{e^{(\tilde{p}_i T_{\nu_o})/(RT_\nu)}+1}
\end{aligned}
$$

At $T = 100$ GK R cancels out T_{ν_o}/T_ν and so

$$
f_\nu(\tilde{p}_i,0) = \frac{1}{e^{\tilde{p}_i}+1}
$$

So this author chooses a discrete grid of values of rescaled p_i on which one can time evolve $f_\nu(\tilde{p}_i,t)$.

The equations for C_{coll} change by

$$
C_{coll} \rightarrow \left(\frac{kT_{\nu_o}}{R}\right)^5 C_{coll}
$$

Note that for the reactions involving electrons and positrons m_e is replaced everywhere with $(m_\mathrm{e}R)/(kT_{\nu_o})$.

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