Second-Order Runge-Kutta Method

$$Y_{n+1}(i) = Y_n(i) + \frac{1}{2}\left[\frac{dY(i)}{dt}(t, Y_n) + \frac{dY(i)}{dt}(t + \Delta t, Y_{n+1})\right]\Delta t \quad (1)$$

$$\frac{dY(i)}{dt} = \sum_{j,k,l} N_i \left(-\frac{Y^{N_i}(i)Y^{N_j}(j)}{N_i!N_j!}[ij] + \frac{Y^{N_k}(k)Y^{N_l}(l)}{N_k!N_l!}[kl]\right)$$

Use Semi-Implicit Euler Method to determing $\dot{Y}$'s for Equation (1)

$$\begin{aligned} Y_{n-1}(i) &= Y_n(i) - \frac{dY(i)}{dt}[t, Y_n]\Delta t_{n-1} \\ \frac{dY(i)}{dt}[t, Y_n] &= \frac{Y_n(i) - Y_{n-1}(i)}{\Delta t_{n-1}} \end{aligned} \quad (2)$$

Linearize the equation for $\dot{Y}$ to get a matrix form.

$$\begin{aligned} \frac{dY(i)}{dt}(t, Y_n) &= \frac{Y_{n\alpha}(i) - Y_{n-1}(i)}{\Delta t_{n-1}} \\ \frac{dY(i)}{dt}(t, Y_n) &= \sum_{i,j,k} -\frac{N_i[ij]}{N_i!N_j!(N_i + N_j)} \\ &\quad [N_i Y_n^{N_i-1}(i)Y_n^{N_j}(j)Y_{n\alpha}(i) + N_j Y_n^{N_i}(i)Y_n^{N_j-1}(j)Y_{n\alpha}(j)] + \frac{N_i[kl]}{N_k!N_l!(N_k + N_l)} \\ &\quad [N_k Y_n^{N_k-1}(k)Y_n^{N_l}(l)Y_{n\alpha}(k) + N_l Y_n^{N_k}(k)Y_n^{N_l-1}(l)Y_{n\alpha}(l)] \\ Y_{n-1}(i) &= Q_{ij}(Y_n, [ij](T_n), [kl](T_n)) \times Y_{n\alpha}(j) \end{aligned}$$