Methods of Calculating the Neutrino Heating Effect in Big Bang Nucleosynthesis

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1 Motivation

I use a Big Bang Nucleosynthesis code to calculate the mass fractions of various isotopes produced by the Big Bang in a homogeneous, isotropic universe, and compare it with experimentally measured amounts. For instance the Helium-4 Mass Fraction has been measured to around $0.232 \pm 0.003 \pm 0.005$. So the third digit is in dispute. But experimentalists hope to resolve that dispute, and want to move on to determining the fourth digit with certainty. In anticipation I calculate this effect which changes $X_{^4He}$ on the order of the fourth digit. But I'll show in this talk that this effect, which I call the MultiBin Model is very complicated and time consuming. In the '80's Tony Rothman and Richard Matzner had an approximate method of calculating the effect quickly, which I call the One Bin Model and outline here as well. Improved computer accuracy and faster CPU time enable me to calculate the MultiBin Model effect and compare its results with those of the One Bin Model. Also the MultiBin Model illustrates the effect in more detail than the One Bin Model.

2 Neutrino Decoupling

During the Big Bang, electron, muon and τ neutrinos are in contact with electrons and positrons via the following reactions.

$$\begin{array}{ll} e^- + \nu_i \leftrightarrow e^- + \nu_e & e^- + \bar{\nu}_i \leftrightarrow e^- + \bar{\nu}_e \\ e^+ + \nu_i \leftrightarrow e^+ + \nu_e & e^+ + \bar{\nu}_i \leftrightarrow e^+ + \bar{\nu}_e \\ e^- + e^+ \leftrightarrow \nu_i + \bar{\nu}_i & \end{array}$$

These are scattering reactions, and the last one is an annihilation one. And they're in contact with each other via these reactions.

$$\begin{array}{ll} \nu_i + \nu_i \leftrightarrow \nu_i + \nu_i & \nu_i + \bar{\nu}_i \leftrightarrow \nu_i + \bar{\nu}_i \\ \nu_i + \nu_j \leftrightarrow \nu_i + \nu_j & \nu_i + \bar{\nu}_j \leftrightarrow \nu_i + \bar{\nu}_j \\ \nu_i + \bar{\nu}_i \leftrightarrow \nu_j + \bar{\nu}_j & \end{array}$$

But these weak reactions have a much slower rate than electromagnetic reactions like

$$e^- + e^+ \leftrightarrow 2\gamma$$

So the effect on the distribution functions of the particles in the weak rates is usually ignored.

But suppose you DON'T want to ignore the effect. Between electron/photon temperature $T = 5.9$ GK and 1 GK e^-e^+ annihilate each other. What if you wanted to find out how much energy from e^-e^+ annihilation is transfered into neutrino pairs instead of photon pairs.

To calculate that effect, I have to modify the equation of energy conservation, which is derived from the stress-energy tensor $T^{ob}{}_{;b}$, for the perfect fluid model of the universe.

$$\begin{aligned}
0 &= \dot{\rho}_{total} + 3\frac{\dot{R}}{R}(\rho_{total} + P_{total}) \\
\dot{\rho}_{e+\gamma} &= -4\frac{\dot{R}}{R}\rho_\gamma - 3\frac{\dot{R}}{R}(\rho_e + P_e) - (\dot{\rho}_\nu + 4\frac{\dot{R}}{R}\rho_\nu) \\
\dot{\rho}_{e+\gamma} &= -4\frac{\dot{R}}{R}\rho_\gamma - 3\frac{\dot{R}}{R}(\rho_e + P_e) - \frac{1}{R^4}\frac{d(\rho_\nu R^4)}{dt} \qquad (1)
\end{aligned}$$

For my code I rewrite this as an equation for the electron/photon energy density $\dot{\rho}_{e+\gamma}$, which I evolve using the 2nd. Order Runge-Kutta method. Neutrino heating puts in Equation (1) a new term that depends on the time derivative of the quantity $\rho_\nu R^4$. ρ_ν is the neutrino energy density while R is the expansion coefficient.

Equation (2) is the equation for ρ_ν. From here on we rescale the momentum variable p to $(pR)/(kT_N)$ ($T_N = 1.946$ K is the present neutrino temperature) so that the Fermi-Dirac $f_{FD}(p) = 1/[\exp(p) + 1]$ ($a = (\pi^2/15)(k^4/(c\hbar)^3)$).

$$\begin{aligned}
\rho_\nu &= \frac{15}{\pi^4}a\left(\frac{T_N}{R}\right)^4 \int dpp^3 f_\nu(p) \qquad (2) \\
\dot{\rho}_\nu &= -4\frac{\dot{R}}{R}\rho_\nu + \frac{15}{\pi^4}a\left(\frac{T_N}{R}\right)^4 \int dpp^3 \dot{f}_\nu(p) \qquad (3)
\end{aligned}$$

Taking the time derivative of (2), I show that the extra term in (1) depends on an integral of p^3 times the time derivative of the neutrino distribution function $f_\nu(p)$. So that time derivative is what I want to calculate in particular.

Equation (4) is the equation for the time evolution of R.

$$\dot{\alpha} = \frac{\dot{R}}{R} = \sqrt{\frac{8}{3}\pi G(\rho_{e+\gamma} + \rho_\nu)} \qquad (4)$$

(I define the quantity $\alpha \equiv \ln(R)$ for future use.) The weak reactions will increase ρ_ν, but take that increase from the electron energy density ρ_e. So the neutrino heating effect doesn't change the evolution of R overall.

The final mass fraction $X_{^4He}$ of Helium-4 depends on the amount of neutrons present at $T \approx 0.8$ GK, when the nucleon reactions are fast enough to produce all the nuclei from the neutrons and protons en masse. X_n in turn depends on the weak reactions that interconvert neutrons and protons. The equations for the conversion rates are shown below.

$$\begin{aligned}
\lambda(n + \nu_e \to p + e^-) &= A\int v_e E_e^2 p_{\nu_e}^2 dp_{\nu_e} f_{\nu_e}(1 - f_e) \\
\lambda(n + e^+ \to p + \bar{\nu_e}) &= A\int p_\nu^2 p_e^2 dp_e f_e(1 - f_{\nu_e}) \\
\lambda(n \to p + e^- + \bar{\nu_e}) &= A\int v_e p_{\nu_e}^2 E_e^2 dp_{\nu_e}(1 - f_{\nu_e})(1 - f_e)
\end{aligned}$$

$$
\begin{aligned}
\lambda(p + e^- \rightarrow n + \nu_e) &= A\int p_{\nu_e}^2 p_e^2 dp_e f_e (1 - f_{\nu_e}) \\
\lambda(p + \bar{\nu}_e \rightarrow n + e^+) &= A\int v_e E_e^2 p_{\nu_e}^2 dp_{\nu_e} f_{\nu_e} (1 - f_e) \\
\lambda(p + e^- + \bar{\nu}_e \rightarrow n) &= A\int v_e p_{\nu_e}^2 E_e^2 dp_{\nu_e} f_{\nu_e} f_e
\end{aligned}
$$

These equations are derived from Fermi's Golden Rule. I use the number densities of the particles involved to get an expression for the density of states. That's how the distribution functions of e and ν_e go into these equations. So any change in f_{ν_e} from neutrino heating can change the conversion rates.

3 The MultiBin Model

Like I've said, I need to calculate the time derivative of f_{ν_e} and f_{ν_μ}. That time derivative obeys the Boltzmann Transport Equation. If we look at particle 1 in the reaction 1+2 $\rightarrow$ 3+4.

$$
\begin{aligned}
\frac{df_1}{dt} &= C_{coll} \qquad (5) \\
C_{coll} &= \frac{1}{2E_1}\left(\frac{kT_N}{R}\right)^5 \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4} S|M|^2 \\
&\quad (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)\Lambda(f_1, f_2, f_3, f_4) \\
\Lambda(f_1, f_2, f_3, f_4) &= \{[1 - f_1(p_1)][1 - f_2(p_2)]f_3(p_3)f_4(p_4) \\
&\quad - f_1(p_1)f_2(p_2)[1 - f_3(p_3)][1 - f_4(p_4)]\}
\end{aligned}
$$

Λ is the phase space factor. The first term in Λ corresponds to the reverse direction of 1+2 $\rightarrow$ 3+4, which creates particle 1. The second term is for the forward direction, which destroys particle one. And the $1 - f$ terms are due to the Fermi-Dirac nature of the particles involved.

Process	$S\|M\|^2$
$\nu_e + \nu_e \to \nu_e + \nu_e$	$32G_F^2(p_1 \cdot p_2)(p_3 \cdot p_4)$
$\nu_e + \nu_i \to \nu_e + \nu_i$	$32G_F^2(p_1 \cdot p_2)(p_3 \cdot p_4)$
$\nu_e + \bar{\nu}_i \to \nu_e + \bar{\nu}_i$	$32G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_e + \bar{\nu}_i \to \nu_i + \bar{\nu}_i$	$32G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_e + \bar{\nu}_e \to \nu_e + \bar{\nu}_e$	$128G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_e + e^- \to \nu_e + e^-$	$32G_F^2[(C_A + C_V)^2(p_1 \cdot p_2)(p_3 \cdot p_4)$ $+(C_A - C_V)^2(p_1 \cdot p_4)(p_2 \cdot p_3)$ $-(C_V^2 - C_A^2)m_e^2(p_1 \cdot p_3)]$
$\nu_e + e^+ \to \nu_e + e^+$	$32G_F^2[(C_A - C_V)^2(p_1 \cdot p_2)(p_3 \cdot p_4)$ $+(C_A + C_V)^2(p_1 \cdot p_4)(p_2 \cdot p_3)$ $-(C_V^2 - C_A^2)m_e^2(p_1 \cdot p_3)]$
$\nu_e + \bar{\nu}_e \to e^- + e^+$	$32G_F^2[(C_A + C_V)^2(p_1 \cdot p_4)(p_2 \cdot p_3)$ $+(C_A - C_V)^2(p_1 \cdot p_3)(p_2 \cdot p_4)$ $+(C_V^2 - C_A^2)m_e^2(p_1 \cdot p_2)]$

Table (1): Matrix Elements for ν_e as Particle # 1.

Process	$S\|M\|^2$
$\nu_i + \nu_i \to \nu_i + \nu_i$	$32G_F^2(p_1 \cdot p_2)(p_3 \cdot p_4)$
$\nu_i + \nu_j \to \nu_i + \nu_j$	$32G_F^2(p_1 \cdot p_2)(p_3 \cdot p_4)$
$\nu_i + \nu_e \to \nu_i + \nu_e$	$32G_F^2(p_1 \cdot p_2)(p_3 \cdot p_4)$
$\nu_i + \bar{\nu}_j \to \nu_i + \bar{\nu}_j$	$32G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_i + \bar{\nu}_e \to \nu_i + \bar{\nu}_e$	$32G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_i + \bar{\nu}_i \to \nu_j + \bar{\nu}_j$	$32G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_i + \bar{\nu}_i \to \nu_e + \bar{\nu}_e$	$32G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_i + \bar{\nu}_i \to \nu_i + \bar{\nu}_i$	$128G_F^2(p_1 \cdot p_4)(p_2 \cdot p_3)$
$\nu_i + e^- \to \nu_i + e^-$	$32G_F^2\{(C_A + C_V - 2)^2(p_1 \cdot p_2)(p_3 \cdot p_4)$ $+(C_A - C_V)^2(p_1 \cdot p_4)(p_2 \cdot p_3)$ $-[(C_V - 1)^2 - (C_A - 1)^2]m_e^2(p_1 \cdot p_3)\}$
$\nu_i + e^+ \to \nu_i + e^+$	$32G_F^2\{(C_A - C_V)^2(p_1 \cdot p_2)(p_3 \cdot p_4)$ $+(C_A + C_V - 2)^2(p_1 \cdot p_4)(p_2 \cdot p_3)$ $-[(C_V - 1)^2 - (C_A - 1)^2]m_e^2(p_1 \cdot p_3)\}$
$\nu_i + \bar{\nu}_i \to e^- + e^+$	$32G_F^2\{(C_A + C_V - 2)^2(p_1 \cdot p_4)(p_2 \cdot p_3)$ $+(C_A - C_V)^2(p_1 \cdot p_3)(p_2 \cdot p_4)$ $+[(C_V - 1)^2 - (C_A - 1)^2]m_e^2(p_1 \cdot p_2)\}$

Table (2): Matrix Elements for ν_μ, ν_τ as Particle # 1.

$S|M|^2$ are the matrix elements corresponding to each different reaction. Tables (1) and (2) list all the reactions that I have to account for in the Boltzmann equation, with Table (1) for the case of ν_e as Particle 1 and Table (2) for the case of ν_μ as Particle 1. I also assume that $f_{\nu_\tau} = f_{\nu_\mu}$. Actually, Hannestad and Madsen showed in a 1996 article that the integrals of C_{coll} can be down to two variables, the magnitudes of p_2 and p_3. For instance, for the reaction $\nu_e + \nu_\mu \leftrightarrow \nu_e + \nu_\mu$, C_{coll} has the final form:

$$
\begin{aligned}
C_{coll} &= \int_0^{\infty} dp_2 \int_0^{p_1+p_2} dp_3 F(p_1,p_2,p_3) \\
&\{[1-f_{\nu_e}(p_1)][1-f_{\nu_\mu}(p_2)] f_{\nu_e}(p_3) f_{\nu_\mu}(p_1+p_2-p_3) \\
&-f_{\nu_e}(p_1) f_{\nu_\mu}(p_2)[1-f_{\nu_e}(p_3)][1-f_{\nu_\mu}(p_1+p_2-p_3)]\}
\end{aligned}
$$

$F(p_1,p_2,p_3)$ is a (very) complicated polynomial of these three momentum magnitudes. The F for reactions involving electrons would involve rescaled electron mass $(m_e R)/(kT_N)$ as well.

In my code, I solve for Equation (5) for 50 set values $p_i = i\Delta p$ of $f_{\nu_e}(p_i)$, and 50 values of $f_{\nu_\mu}(p_i)$. So I have a total of 100 bins for f_ν. Since I know $f_{\nu_e}(p_i)$ and $f_{\nu_\mu}(p_i)$ for only a finite number of p_i I have to use the extended trapezoidal rule to write sums corresponding to the integrals. For instance:

$$
\begin{aligned}
C_{coll} &= \sum_0^{i_{2Max}} \sum_0^{i_1+i_2} (\Delta p)^2 C_1 C_2 F(p_1,p_2,p_3) \\
&\{[1-f_{\nu_e}(p_1)][1-f_{\nu_\mu}(p_2)] f_{\nu_e}(p_3) f_{\nu_\mu}(p_1+p_2-p_3) \\
&-f_{\nu_e}(p_1) f_{\nu_\mu}(p_2)[1-f_{\nu_e}(p_3)][1-f_{\nu_\mu}(p_1+p_2-p_3)]\}
\end{aligned}
$$

C_1 and $C_2 = 0$ at the endpoints of the sums but 1 everywhere else. But for the highest temperatures, Λ is a very small difference of two very large numbers. Like in the equation for the time derivatives of isotope abundances Y_i

$$
\frac{dY(i)}{dt} = \sum_{j,k,l} N_i \left(-\frac{Y^{N_i}(i) Y^{N_j}(j)}{N_i! N_j!}[ij] + \frac{Y^{N_k}(k) Y^{N_l}(l)}{N_k! N_l!}[kl] \right)
$$

So the strategy I'm taking is to solve for the 100 bins of f_ν the same way I solve for the 68 isotopes. For timestep n I multiply out Λ and average C_{coll} between the bins of n and $n-1$, getting the expression:

$$
\begin{aligned}
C_{coll} &= \frac{f_{\nu_e}(p_1,t_n) - f_{\nu_e}(p_1,t_{n-1})}{\Delta t_{n-1}} \\
C_{coll} &= \sum_0^{i_{2Max}} \sum_0^{i_1+i_2} (\Delta p)^2 C_1 C_2 F(p_1,p_2,p_3) \\
&\left\{ \frac{1}{2}[f_{\nu_e}(p_3,t_n) f_{\nu_\mu}(p_4,t_{n-1}) + f_{\nu_e}(p_3,t_{n-1}) f_{\nu_\mu}(p_4,t_n)] \right. \\
&-\frac{1}{2}[f_{\nu_e}(p_1,t_n) f_{\nu_\mu}(p_2,t_{n-1}) + f_{\nu_e}(p_1,t_{n-1}) f_{\nu_\mu}(p_2,t_n)] \\
&-\frac{1}{3}[f_{\nu_e}(p_1,t_n) f_{\nu_e}(p_3,t_{n-1}) f_{\nu_\mu}(p_4,t_{n-1}) + f_{\nu_e}(p_1,t_{n-1}) f_{\nu_e}(p_3,t_n) f_{\nu_\mu}(p_4,t_{n-1}) \\
&+f_{\nu_e}(p_1,t_{n-1}) f_{\nu_e}(p_3,t_{n-1}) f_{\nu_\mu}(p_4,t_n)] - \frac{1}{3}[f_{\nu_\mu}(p_2,t_n) f_{\nu_e}(p_3,t_{n-1}) f_{\nu_\mu}(p_4,t_{n-1}) \\
&+f_{\nu_\mu}(p_2,t_{n-1}) f_{\nu_e}(p_3,t_n) f_{\nu_\mu}(p_4,t_{n-1}) + f_{\nu_\mu}(p_2,t_{n-1}) f_{\nu_e}(p_3,t_{n-1}) f_{\nu_\mu}(p_4,t_n)]
\end{aligned}
$$

$$+\frac{1}{3}[f_{\nu_e}(p_1,t_n)f_{\nu_\mu}(p_2,t_{n-1})f_{\nu_e}(p_3,t_{n-1})+f_{\nu_e}(p_1,t_{n-1})f_{\nu_\mu}(p_2,t_n)f_{\nu_e}(p_3,t_{n-1})$$
$$+f_{\nu_e}(p_1,t_{n-1})f_{\nu_\mu}(p_2,t_{n-1})f_{\nu_e}(p_3,t_n)]+\frac{1}{3}[f_{\nu_e}(p_1,t_n)f_{\nu_\mu}(p_2,t_{n-1})f_{\nu_\mu}(p_4,t_{n-1})$$
$$+f_{\nu_e}(p_1,t_{n-1})f_{\nu_\mu}(p_2,t_n)f_{\nu_\mu}(p_4,t_{n-1})+f_{\nu_e}(p_1,t_{n-1})f_{\nu_\mu}(p_2,t_{n-1})f_{\nu_\mu}(p_4,t_n)]\Big\}$$

which we can rewrite as a matrix equation to solve for $f_\nu(p_i,t_n)$ and plug that back into $\dot{f}_\nu(p_i,t_n)$. This is how I go about calculating a final value for $f_\nu(p_i)$ and $\dot{f}_\nu(p_i)$ via 2nd. Order Runge-Kutta. Then I just plug in these values into the equations for $\dot{\rho}_\nu$ and λ.

4 The One Bin Model

To avoid the Boltzmann equation and the distribution functions altogether, Rothman and Matzner in a 1984 article calculated the effect by assuming that for all values of p, $f_\nu(p)$ is increased by the same proportional amount. Hence I use the name the One Bin Model.

Let's say that at a time t, $f_\nu(p)$ has had energy transferred to it from

$$f_e(p,R') = \frac{1}{e^{\frac{p}{T}\frac{T_N}{R}}+1}$$

of the e^+e^- pairs, taken at the **same** value of p from all time $t'<t$. The contribution from t' is determined by a weight function $W_\nu(\alpha';\alpha)$. Then RM defines f_ν as:

$$f_\nu(p,\alpha) = \int_{-\infty}^{\alpha} d\alpha' \frac{W_\nu(\alpha';\alpha)}{e^{\frac{p}{T}\frac{T_N}{R}}+1}$$

and plug that into Equation (2). Integrating for p

$$\begin{aligned}
\rho_\nu &= \frac{15}{\pi^4} a \left(\frac{T_N}{R}\right)^4 \int dp p^3 \int_{-\infty}^{\alpha} d\alpha' \frac{W_\nu(\alpha';\alpha)}{e^{\frac{p}{T}\frac{T_N}{R}}+1} \\
\rho_\nu &= \frac{7}{8} a \left(\frac{T_N}{R}\right)^4 \int_{-\infty}^{\alpha} d\alpha' \left(R'\frac{T}{T_N}\right)^4 W_\nu(\alpha';\alpha) \qquad (6)\\
\rho_\nu &= \rho_{\nu 0} \int_{-\infty}^{\alpha} d\alpha' \left(R'\frac{T}{T_N}\right)^4 W_\nu(\alpha';\alpha) \\
\rho_\nu R^4 &= \frac{7}{8} a T_N^4 \int_{-\infty}^{\alpha} d\alpha' \left(R'\frac{T}{T_N}\right)^4 W_\nu(\alpha';\alpha)
\end{aligned}$$

shows that $\rho_\nu =$ the ρ_{ν_0} for the case of no neutrino heating times a function that depends only on time, or that $\rho_\nu R^4 =$ this function that depends only on time, illustrating the fact that $f_\nu(p)$ is heated equally for all p. One can derive a similar time-only dependent function for the quantity $n_\nu R^3$ where n_ν is the neutrino number density.

$$n_\nu = \frac{15}{\pi^4}\frac{a}{k}\left(\frac{T_N}{R}\right)^3 \int dp p^2 f_\nu(p)$$

$$
\begin{aligned}
n_\nu &= \frac{15}{\pi^4}\frac{a}{k}\left(\frac{T_N}{R}\right)^3 (1.803)\int_{-\infty}^{\alpha} d\alpha' \left(R'\frac{T}{T_N}\right)^3 W_\nu(\alpha';\alpha) \\
n_\nu R^3 &= \frac{15}{\pi^4}\frac{a}{k} T_N^3 (1.803)\int_{-\infty}^{\alpha} d\alpha' \left(R'\frac{T}{T_N}\right)^3 W_\nu(\alpha';\alpha)
\end{aligned}
$$

Rothman and Matzner look at only the average times of the reactions listed in Tables (1) and (2). So they assume that $\rho_\nu R^4$ and $n_\nu R^3$ are decayed exponentially by reactions that deplete ν. So for example RM constructs an equation for the time derivative of $n_{\nu_e} R^3$

$$
\begin{aligned}
\frac{d}{dt}(n_{\nu_e} R^3) &= -\left(\frac{n_{\nu_e} R^3}{t(\nu_e\bar{\nu}_e \to e^- e^+)} + \frac{n_{\nu_e} R^3}{t(\nu_e\bar{\nu}_e \to \nu_\mu\bar{\nu}_\mu)} + \frac{n_{\nu_e} R^3}{t(\nu_e\bar{\nu}_e \to \nu_\tau\bar{\nu}_\tau)}\right) \\
&\quad + \frac{n_{\nu_\mu} R^3}{t(\nu_\mu\bar{\nu}_\mu \to \nu_e\bar{\nu}_e)} + \frac{n_{\nu_\tau} R^3}{t(\nu_\tau\bar{\nu}_\tau \to \nu_e\bar{\nu}_e)} + \frac{n_e R^3}{t(e^- e^+ \to \nu_e\bar{\nu}_e)}
\end{aligned} \tag{7}
$$

that equals minus $n_{\nu_e} R^3$ divided by the reaction times that destroy $\nu_e\bar{\nu}_e$ pairs, plus $n_{\nu_\mu} R^3$ divided by the reaction time that destroys $\nu_\mu\bar{\nu}_\mu$ pairs and creates $\nu_e\bar{\nu}_e$ pairs, and the same for $n_{\nu_\tau} R^3$ and $n_e R^3$. They also compose similar equations for the time derivatives of $n_{\nu_\mu} R^3$ and $n_{\nu_\tau} R^3$.

Similarly for $\rho_{\nu_e} R^4$

$$
\frac{d}{dt}(\rho_{\nu_e} R^4) = -\frac{\rho_{\nu_e} R^4}{t(\nu_e \to)} + \frac{\rho_{\nu_e} R^4}{t(\nu_e \to \nu_e)} + \frac{\rho_{\nu_\mu} R^4}{t(\nu_\mu \to \nu_e)} + \frac{\rho_{\nu_\tau} R^4}{t(\nu_\tau \to \nu_e)} + \frac{\rho_e R^4}{t(e \to \nu_e)} \tag{8}
$$

$t(\nu_e \to)$ refers to the reaction times with ν_e as an incoming particle:

$$
\begin{aligned}
\frac{1}{t(\nu_e \to)} &= \frac{1}{t(\nu_e\nu_e \to \nu_e\nu_e)} + \frac{1}{t(\nu_e\bar{\nu}_e \to \nu_e\bar{\nu}_e)} + \frac{1}{t(\nu_e\nu_\mu \to \nu_e\nu_\mu)} \\
&\quad + \frac{1}{t(\nu_e\bar{\nu}_\mu \to \nu_e\bar{\nu}_\mu)} + \frac{1}{t(\nu_e\nu_\tau \to \nu_e\nu_\tau)} + \frac{1}{t(\nu_e\bar{\nu}_\tau \to \nu_e\bar{\nu}_\tau)} \\
&\quad + \frac{1}{t(\nu_e\bar{\nu}_e \to \nu_\mu\bar{\nu}_\mu)} + \frac{1}{t(\nu_e\bar{\nu}_e \to \nu_\tau\bar{\nu}_\tau)} + \frac{1}{t(\nu_e e \to \nu_e e)} \\
&\quad + \frac{1}{t(\bar{\nu}_e e \to \bar{\nu}_e e)} + \frac{1}{t(\nu_e\bar{\nu}_e \to e^- e^+)}
\end{aligned}
$$

while $t(\nu_\mu \to \nu_e)$ accounts for reactions that have ν_μ as an incoming particle and ν_e as an outgoing one.

$$
\frac{1}{t(\nu\mu \to \nu_e)} = \frac{1}{2t(\nu_\mu\nu_e \to \nu_\mu\nu_e)} + \frac{1}{2t(\nu_\mu\bar{\nu}_e \to \nu_\mu\bar{\nu}_e)} + \frac{1}{2t(\nu_\mu\bar{\nu}_\mu \to \nu_e\bar{\nu}_e)}
$$

(The factor of 2 indicates that the incoming energy is evenly distributed between the outgoing particles.) Reaction times are determined from average cross sections by the standard formula $1/t = n_\nu \sigma c$. The One Bin Model then uses these time derivatives to evolve $\rho_\nu R^4$ and $n_\nu R^3$ by second order Runge-Kutta.

For the neutrinos RM define a central temperature $T_{\nu c}$ such that

$$
\begin{aligned}
T_{\nu c} &= \frac{\rho_\nu}{n_\nu} \frac{\frac{15}{\pi^4}\frac{a}{k}(1.803)}{\frac{7}{8}a} \\
T_{\nu c} &= \frac{T_N}{R} \frac{\int_{-\infty}^{\alpha} d\alpha' \left(R' \frac{T}{T_N}\right)^4 W_\nu(\alpha';\alpha)}{\int_{-\infty}^{\alpha} d\alpha' \left(R' \frac{T}{T_N}\right)^3 W_\nu(\alpha';\alpha)}
\end{aligned}
$$

RM use these $T_{\nu c}$ to calculate σ. They also define f_{ν_e}
(rescaled p) for the electron neutrinos as

$$
f_{\nu_e}(p) = \frac{n_{\nu_e}}{n_{\nu 0}(T_{\nu_e c})} \frac{1}{e^{\frac{p}{R}\frac{T_N}{T_{\nu_e c}}} + 1} \tag{9}
$$

$n_{\nu 0}$ is the neutrino number density for the case without neutrino heating. Plugged into the equation for n_{ν_e} the above f_{ν_e} would give the correct expression for n_{ν_e} with neutrino heating. But to be correct, one has to calculate $n_{\nu 0}$ for the central electron neutrino temperature, which is different from the T_ν it would have without heating. Since the conversion rates use n_{ν_e} for their density of states RM uses Equation (9) in the conversion rates as well.

5 Final Results

Figure (1) graphs the fractional increases in ρ_{ν_e} and ρ_{ν_μ} for both the One Bin and the MultiBin model, and in both cases the MultiBin model has the higher value. Figure (2) shows the increase of $f_{\nu_e}(p_i)$ from the Fermi-Dirac value $f_{FD}(p_i)$ for $p_i = 1$ to 25. For the One Bin model, Equation (6) is equivalent to multiplying the $f_\nu(p)$ by

$$
\begin{aligned}
f_{\nu_i}(p) &\rightarrow f_\nu(p) \int_{-\infty}^{\alpha} d\alpha' \left(R' \frac{T}{T_N}\right)^3 W_\nu(\alpha';\alpha) \\
f_{\nu_i}(p) &\rightarrow f_{\nu_i}(p) \frac{\rho_{\nu_i}}{\rho_{\nu 0}}
\end{aligned}
$$

This point on Figure (2) marks the amount all $f_{\nu_e}(p)$ in the One Bin model is heated. Only a few of the bins have an increase equal to or less than the One Bin change. The higher energy bins have a much larger energy transfer from the $e^\pm$ pairs, raising ρ_{ν_e} over the One Bin result. But note that the f/f_{FD} ratios reach a final value, once the rates involving that $f_{\nu_e}(p_i)$ are no longer effective. At T slightly over 1 GK all the bins large enough to affect the value of ρ_{ν_e} have reached their final values, and so ρ_{ν_e} flattens out.

For higher energy bins the reactions stay effective longer. But near $T = 1$ GK the higher $f_{\nu_e}(p_i)$'s have a sudden decrease. Possibly, the $e^- e^+$ annihilation rate $e^- e^+ \rightarrow \nu_e \bar{\nu}_e$ has ceased to be effective then. But rates that deplete the higher energy neutrinos into lower energy neutrinos are still working. So $f_{\nu_e}(p_i)$ decreases until these destruction rates themselves are no longer effective. The situation is analogous to the deuterium mass fraction X_d. I think, though that bins higher than 25 would replenish the decreasing bins, reducing the decrease. So this decrease might be an indicator of my BBN code's limitations.

f_{ν_μ} has similar behavior, but much less increase. Also, its highest bins don't get nearly high enough to undergo depletion.

The ρ_{ν_i} increase. Therefore the ρ_e decrease. Due to this decrease, at a given time t temperature T for the models is smaller than T without neutrino heating. So Figures (3) and (4) are graphed vs. the time rather than T. T for the case without heating is shown on the top border.

Figure (3) shows $X_{^4He}$ (the differences between the models is imperceptable on the graph's scale) and $\Delta X_{^4He}(OB, MB) = X_{^4He}(OB, MB) - X_{^4He}$. For most of the run, $X_{^4He}(OB)$ is slightly smaller than $X_{^4He}(MB)$ (too small to see on the graph) perhaps because of the One Bin model's smaller ρ_ν's and slightly larger T. But just after Nucleosynthesis at around $T = 7$ GK $X_{^4He}(OB)$ surpasses $X_{^4He}(MB)$ leading to a larger final result.

$$\begin{aligned} X_{^4He} &= 0.239272 \\ X_{^4He}(OB) &= 0.239736 \\ X_{^4He}(MB) &= 0.239432 \end{aligned}$$

$\Delta X_{^4He}(OB)$ seems to flatten out earlier than $\Delta X_{^4He}(MB)$. Perhaps the higher energy bins in the MultiBin model convert more neutrons to protons than in the One Bin model.

Figure (4) shows $\lambda(n \to p)$ and $\lambda(p \to n)$ (Again, the differences between models are imperceptable.) and $\Delta\lambda(n \to p, OB, MB) = \lambda(n \to p, OB, MB) - \lambda(n \to p)$, $\Delta\lambda(p \to n, OB, MB) = \lambda(p \to n, OB, MB) - \lambda(p \to n)$. The timestep size makes quantities difficult to calculate at the highest temperatures. So $\Delta\lambda$ is initially negative in the One Bin model while it varies considerably in the MultiBin model. But eventually both models increase their λ's. Moreso for the One Bin model at first. But the increases of $\lambda(n \to p, OB)$ and $\lambda(p \to n, OB)$ are more equal than for the MultiBin model. Eventually $\lambda(n \to p, OB)$ falls below $\lambda(n \to p, MB)$, and is less at the time when Nucleosynthesis occurs. A lower rate means fewer neutrons have decayed. But this difference could be too small to fully account for the higher $X^{^4He}$ in the One Bin model.

6 Future Research

The One Bin model calculates too low an average for the integrals of ρ_{ν_i}. Perhaps one can modify the way the model calculates reaction times so that the model will give an average that always agrees with the MultiBin results. Also, I want to quantitatively establish exactly how much $\Delta\lambda$ changes $\Delta X_{^4He}$ and when Nucleosynthesis in each model occurs, to determine with certainity why the One Bin model has a higher $X_{^4He}$ at the end. I'll also look for better ways to determine the timestep so that the graphs of physical quantities will be smooth at the highest temperatures. The number of bins can be increased and extended to higher values of p_i. I believe that those modifications will solve the Boltzmann equation more accurately and reduce the depletion of the current highest bins. Then one can apply the MultiBin model to anisotropic or inhomogeneous models of the universe.

7 References

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