Reciprocal Relativity of Non-Inertial Frames: Quantum Mechanics

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Preface: Inertial frames
Special relativistic quantum mechanics

The Lorentz group defines transformations between inertial frames ($d\,p/d\,t = 0$) of special relativity with invariant Minkowski line element.

In the classical limit (group contraction $c \to \infty$), the Euclidean group defines Newtonian relativity with invariant Newtonian time line element. (Inhomogeneous Euclidean group is Galilei group.)

Quantum states are represented as rays in a Hilbert space

Due to work of Wigner, special relativistic quantum mechanics is understood in terms of the projective representations of the inhomogeneous Lorentz group.

Projective representations of a group are equivalent to the unitary representations of its central extension. The central extension may be algebraic and/or the topological cover. For inhomogeneous Lorentz group, the central extension is the cover; the Poincaré group $\mathcal{SL}(2, \mathbb{C}) \otimes_s \mathcal{T}(4)$.

Single particle wave equations (Klein-Gordon, Dirac, Weyl, Maxwell,...) arise from the Hermitian representations of the Casimir invariants in the enveloping algebra of the Poincaré group. The unitary irreducible representations are labeled by mass and spin.

Outline of talk

1. Classical cases

   Euclidean relativity group for inertial frames

   Hamilton relativity group for noninertial frames

2. Relativistic cases

   Lorentz relativity group for inertial frames in special relativity

   New physical assumption: Born-Green metric and constant *b*

   Unitary relativity group for noninertial frames in reciprical relativity

   - Lorentz group as inertial special case

   - Hamilton group as classical limit

3. Quantum theory: Projective representations as the inhomogeneous unitary group

   Central extension is the quapectic group with Weyl-Heisenberg group as normal subgroup

   Unitary irreducible representations determined from Mackey theorems for semidirect product groups

   Wave equations from representations of the Casimir invariants. Scalar is relativistic oscillator

Euclidean relativity group for Newtonian inertial frames

Consider Newtonian spacetime $x \in \mathbb{M} \simeq \mathbb{R}^{n+1}$, $x = (q, t)$, $q \in \mathbb{R}^n$, $t \in \mathbb{R}$, $n = 3$ the physical case. Co-tangent space $T^*{}_x\mathbb{M}$ spanned by forms $d\,x = (d\,q,\ d\,t)$.

$\Lambda \in \mathcal{GL}(n+1, \mathbb{R})$ acts on $T^*{}_x\mathbb{M} : d\,\tilde{x} = \Lambda \cdot d\,x$. Λ realized by $(n+1)\times(n+1)$ dimensional matrices.

Invariance of Newtonian line element is $d\,t^2$ restricts $\Lambda \in \mathcal{I}\,\hat{\mathcal{G}}\,\mathcal{L}(n, \mathbb{R})$. Additionally, invariance of length $d\,q^2$ in rest frame restrict $\Lambda \in \hat{\mathcal{E}}(n)$.

Newtonian line element is $d\,s^2 = d\,t^2 = {}^t d\,x \cdot \eta^\circ \cdot d\,x = \eta^\circ{}_{a\,b}\, d\,x^a\, d\,x^b$.

Subgroup leaving $d\,s^2$ invariant has property that with R a $n\times n$ submatrix, $v, w \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}$

$$ {}^t\Lambda \cdot \eta^\circ \cdot \Lambda = \eta^\circ \quad \text{or} \quad \eta^\circ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} {}^tR & {}^tw \\ {}^tv & \epsilon \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & v \\ w & \epsilon \end{pmatrix} = \begin{pmatrix} {}^tw\,w & {}^tw\,\epsilon \\ \epsilon\,w & \epsilon^2 \end{pmatrix} $$

and so $w = 0$, $\epsilon = \pm 1$, $\Lambda(\epsilon, R, v) = \begin{pmatrix} R & v \\ 0 & \epsilon \end{pmatrix}$. Group multiplication and inverse are realized by matrix multiplication and matrix inverse

$$ \Lambda(\epsilon, R, v) = \Lambda(\epsilon'', R'', v'') \cdot \Lambda(\epsilon', R', v') = \Lambda(\epsilon''\,\epsilon', R'' \cdot R', R'' \cdot v' + \epsilon'\,v'') $$
$$ \Lambda^{-1}(\epsilon, R, v) = \Lambda(\epsilon, R^{-1}, -\epsilon\,R^{-1} \cdot v) $$

Euclidean relativity group: invariance of length

Asserted that this is the group $\mathcal{I}\hat{\mathcal{G}}\mathcal{L}(n, \mathbb{R}) = \mathcal{D}_2 \otimes_s \mathcal{GL}(n, \mathbb{R}) \otimes_s \mathcal{T}(n)$

$\Lambda(1, I_n, v) \in \mathcal{T}(n) \simeq (\mathbb{R}^n, +)$ translation group and is a normal subgroup as invariant under automorphisms

$$\Lambda(\epsilon', R', v') \cdot \Lambda(1, I_n, v) \cdot \Lambda^{-1}(\epsilon', R', v') = \Lambda(1, I_n, \epsilon' R' \cdot v)$$

Then as $\Lambda(1, R, 0) \in \mathcal{GL}(n, \mathbb{R})$ and $\Lambda(\epsilon, \mathrm{I}_n, 0) \in \mathcal{D}_2$, the group is the extended inhomogeneous linear group.

$$\Lambda(\epsilon, R, v) \in \mathcal{I}\hat{\mathcal{G}}\mathcal{L}(n, \mathbb{R}) \simeq \mathcal{D}_2 \otimes_s \mathcal{GL}(n, \mathbb{R}) \otimes_s \mathcal{T}(n)$$

Next, invariance of length $d q^2 = {}^t d x \cdot \eta^q \cdot d x$ in the inertial rest frame restricts $\Gamma \in \hat{\mathcal{E}}(n)$

${}^t\Lambda(\epsilon, R, 0) \cdot \eta^q \cdot \Lambda(\epsilon, R, 0) = \eta^q$ or

$$\eta^q = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} {}^tR & 0 \\ {}^tv & \epsilon \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R & v \\ 0 & \epsilon \end{pmatrix} = \begin{pmatrix} {}^tR \cdot R & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore $R \in \mathcal{O}(n)$ and therefore the group is the extended Euclidean group

$$\hat{\mathcal{E}}(n) \simeq \mathcal{D}_2 \otimes_s \mathcal{O}(n) \otimes_s \mathcal{T}(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{SO}(n) \otimes_s \mathcal{T}(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{E}(n)$$

Euclidean relativity group: Free particle inertial motion

Euclidean action on basis is $d\,\tilde{x} = \Lambda \cdot d\,x$:

$$d\,\tilde{q} = R \cdot d\,q + v\,d\,t$$
$$d\,\tilde{t} = d\,t$$

Finally, consider diffeomorphisms $\varphi : \mathbb{M} \to \mathbb{M} : x \mapsto \tilde{x} = \varphi(x)$ that leave metrics invariant. Jacobian must be an element Λ of the group. In particular for $\Lambda(1, I_n, v)$

$$\left[\frac{\partial \varphi^a(x)}{\partial x^a}\right] = \begin{pmatrix} \frac{\partial \varphi^i(t,q)}{\partial q^j} & \frac{\partial \varphi^i(t,q)}{\partial t} \\ \frac{\partial \varphi^0(t,q)}{\partial q^j} & \frac{\partial \varphi^0(t,q)}{\partial t} \end{pmatrix} = \Lambda(1, I_n, v) = \begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix}$$

and so for v constant, integrates to the equation for free particle inertial motion

$$\tilde{q}^i = \varphi^i(q, t) = q^i + v^i\,t$$
$$\tilde{t} = \varphi^0(q, t) = t$$

Velocity addition $v = v' + v''$ is given by group product:

$$\Lambda(1, I_n, v') \cdot \Lambda(1, I_n, v'') = \Lambda(1, I_n, v) = \Lambda(1, I_n, v' + v'')$$

Hamilton relativity group for noninertial frames: $d\, p / d\, t \neq 0$

Consider momentum, position, energy, time space $z \in \mathbb{P} \simeq \mathbb{R}^{2n+2}$, $z = (p, q, e, t)$, $p, q \in \mathbb{R}^n$, $t, e \in \mathbb{R}$, $n = 3$ the physical case. Co-tangent space $T^*{}_z \mathbb{P}$ spanned by forms $d\, z = (d\, p, d\, q, d\, e, d\, t)$.

Consider the invariance of Newtonian time line element $d\, s^2 = d\, t^2$ and the length $d\, q^2$ in the inertial rest frame as in the Euclidean case.

Consider also the invariance the symplectic metric required by Hamilton mechanics on this space.

$$ {}^t d\, z \cdot \zeta \cdot d\, z = -d\, e \wedge d\, t + \delta_{i,j}\, d\, p^i \wedge d\, q^j, \quad i, j = 1, \ldots n $$

$\Phi \in \mathcal{GL}(2n+2, \mathbb{R})$ acts on $T^*{}_z \mathbb{P} : d\, \tilde{z} = \Phi \cdot d\, z$. Φ are $(2n+2) \times (2n+2)$ dimensional matrices.

Invariance of the Newtonian line element restricts group to $\mathcal{I}\,\hat{\mathcal{G}}\,\mathcal{L}(2n+1, \mathbb{R})$.

Invariance of symplectic metric restricts general linear group to the symplectic group $\mathcal{S}p(2n+2)$.

The group with invariant Newtonian line element and symplectic metric is the intersection:

$$ \mathcal{S}p(2n+2) \bigcap \mathcal{I}\,\hat{\mathcal{G}}\,\mathcal{L}(2n+1) \simeq \mathcal{H}\,\hat{\mathcal{S}}\,p(2n) = \mathcal{D}_2 \otimes_s \mathcal{S}p(2n) \otimes_s \mathcal{H}(n) $$

where $\mathcal{H}(n)$ is the Weyl-Heisenberg group.

Hamilton relativity: Symplectic invariance

After requiring invariance of $d\,t^2$ it follows from ${}^t\Phi\cdot\zeta\cdot\Phi=\zeta$,

$$\begin{pmatrix} {}^tA & c & 0 \\ {}^tb & a & 0 \\ {}^tw & r & \epsilon \end{pmatrix}\cdot\begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} A & b & w \\ {}^tc & a & r \\ 0 & 0 & \epsilon \end{pmatrix}=\begin{pmatrix} \zeta^\circ & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $\zeta^0=\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ and A is a $(2\,n)\times(2\,n)$ submatrix, $c,\ b,\ w\in\mathbb{R}^{2\,n}$ and $a,\ r\in\mathbb{R}$, $\epsilon=\pm1$.

Solving this results in ${}^tA\cdot\zeta^\circ\cdot A=\zeta^\circ$ and therefore $A\in\mathcal{S}p(2\,n)$. Also $b=0$, $a=\epsilon$ and $c=\epsilon\,\zeta^\circ\cdot A^{-1}\cdot w$. Therefore

$$\Phi(\epsilon,\ A,\ w,\ r)\simeq\begin{pmatrix} A & 0 & w \\ -\epsilon\ {}^tw\cdot\zeta^\circ\cdot A & \epsilon & r \\ 0 & 0 & \epsilon \end{pmatrix}$$

Then $\Phi(\epsilon,\ I_n,\ 0,\ 0)\in\mathcal{D}_2\otimes\mathcal{S}p(2\,n)$, $\quad \mathrm{H}(w,\ r)=\Phi(1,\ I_n,\ w,\ r)\in\mathcal{H}(n)$

$$\Phi(\epsilon,\ A,\ 0,\ 0)\simeq\begin{pmatrix} A & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & \epsilon \end{pmatrix},\ \mathrm{H}(w,\ r)\simeq\begin{pmatrix} I_{2\,n} & 0 & w \\ -\ {}^tw\cdot\zeta^\circ & 1 & r \\ 0 & 0 & 1 \end{pmatrix}\text{ or }\ \mathrm{H}(f,\ v,\ r)\simeq\begin{pmatrix} I_n & 0 & 0 & f \\ 0 & I_n & 0 & v \\ v & -f & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $w\in\mathbb{R}^{2\,n}$ is written as $w=(f,\ v)$, $f,\ v\in\mathbb{R}^n$.

Hamilton relativity: Group operations

Group multiplication and inverse determined from matrix multiplication and inverse

$$\Phi(\epsilon, A, w, r) = \Phi(\epsilon'', A'', w'', r'') \cdot \Phi(\epsilon', A', w', r')$$
$$= \Phi(\epsilon'' \epsilon', A'' \cdot A', A'^{-1} \cdot w'' + \epsilon' w', r' + \epsilon' r'' + \epsilon'' \ {}^t w'' \cdot \zeta^\circ \cdot A' \cdot w')$$

$$\Phi^{-1}(\epsilon, A, w, r) = \Phi(\epsilon, A^{-1}, -\epsilon A^{-1} \cdot w, - r)$$

Group operations of the Weyl-Heisenberg subgroup $\mathrm{H}(w, r) = \Phi(1, I, w, r)$ are

$$\mathrm{H}(w'', r'') \cdot \mathrm{H}(w', r') = \mathrm{H}(w, r) = \mathrm{H}(w'' + w', r'' + r' + {}^t w'' \cdot \zeta^\circ \cdot w')$$

$$\mathrm{H}^{-1}(w, r) = \mathrm{H}(-w, -r)$$

Automorphisms of the subgroup

$$\Phi(\epsilon, A, w, r) = \Phi(\epsilon', A', w', r') \cdot \mathrm{H}(w'', r'') \cdot \Phi^{-1}(\epsilon', A', w', r')$$
$$= \mathrm{H}(\epsilon' A' \cdot w'', r'' - {}^t w' \cdot \zeta^\circ \cdot (A' \cdot w'') + {}^t (A' \cdot w'') \cdot \zeta^\circ \cdot w')$$

Therefore the subgroup $\mathrm{H}(w, r)$ is normal and the group structure is

$$\mathcal{H} \hat{S} p(n) \simeq \mathcal{D}_2 \otimes_s \mathcal{S}p(2 n) \otimes_s \mathcal{H}(n)$$

Hamilton relativity: Weyl-Heisenberg group

$w \in \mathbb{R}^{2n}$ may be written as $w = (f, v)$, $f, v \in \mathbb{R}^n$ and the group operations are then

$$\mathrm{H}(f'', v'', r'') \cdot \mathrm{H}(f', v', r') = \mathrm{H}(f, v, r) = \mathrm{H}(f'' + f', v'' + v', r'' + r' + f'' \cdot v' - v'' \cdot f')$$

$$\mathrm{H}^{-1}(f, v, r) = \mathrm{H}(-f, -v - r)$$

Weyl-Heisenberg group is the semidirect product $\mathcal{H}(n) \simeq \mathcal{T}(n) \otimes_s \mathcal{T}(n+1)$

$$\mathrm{H}(0, v'', 0) \cdot \mathrm{H}(0, v', 0) = \mathrm{H}(f, v, r) = \mathrm{H}(0, v'' + v', 0), \;\; \mathrm{H}^{-1}(0, v, 0) = \mathrm{H}(0, -v, 0)$$

$$\mathrm{H}(f'', 0, r'') \cdot \mathrm{H}(f', 0, r') = \mathrm{H}(f, v, r) = \mathrm{H}(f'' + f', 0, r'' + r'), \;\; \mathrm{H}^{-1}(f, 0, r) = \mathrm{H}(-f, 0, -r)$$

$$\mathrm{H}(0, v, 0) \cdot \mathrm{H}(f, 0, r) \cdot \mathrm{H}^{-1}(0, v, 0) = \mathrm{H}(f, 0, r - 2 f \cdot v)$$

Action of $\mathrm{H}(f, v, r) \in \mathcal{H}(n)$ on a basis is $d\tilde{z} = \mathrm{H} \cdot dz$

$$\begin{aligned} d\tilde{t} &= dt \\ d\tilde{p} &= dp + f\, dt \\ d\tilde{q} &= dq + v\, dt \\ d\tilde{e} &= de + v \cdot dp - f \cdot dq + r\, dt \end{aligned}$$

Addition of velocity v, force f and power r given by the group multiplication law.

Hamilton relativity: Hamilton's equations

Consider the diffeomorphisms $\varphi : \mathbb{P} \to \mathbb{P} : z \mapsto \tilde{z} = \varphi(z)$. Then $\left[\frac{\partial \varphi(z)}{\partial z} \right] = \Phi \in \mathcal{HSp}(2\,n)$ and in particular $\mathcal{H}(n)$

$$\left[\frac{\partial \varphi^\alpha(z)}{\partial z^\beta} \right] = \begin{pmatrix} \frac{\partial \varphi^a(z)}{\partial p^b} & \frac{\partial \varphi^a(z)}{\partial q^b} & \frac{\partial \varphi^a(z)}{\partial e} & \frac{\partial \varphi^a(z)}{\partial t} \\ \frac{\partial \varphi^{n+a}(z)}{\partial p^b} & \frac{\partial \varphi^{n+a}(z)}{\partial q^b} & \frac{\partial \varphi^{n+a}(z)}{\partial e} & \frac{\partial \varphi^{n+a}(z)}{\partial t} \\ \frac{\partial \varphi^{2\,n+1}(z)}{\partial p^b} & \frac{\partial \varphi^{2\,n+1}(z)}{\partial q^b} & \frac{\partial \varphi^{2\,n+1}(z)}{\partial e} & \frac{\partial \varphi^{2\,n+1}(z)}{\partial t} \\ \frac{\partial \varphi^{2\,n+2}(z)}{\partial p^b} & \frac{\partial \varphi^{2\,n+2}(z)}{\partial q^b} & \frac{\partial \varphi^{2\,n+2}(z)}{\partial e} & \frac{\partial \varphi^{2\,n+2}(z)}{\partial t} \end{pmatrix} = \mathrm{H} = \begin{pmatrix} \delta_{a\,b} & 0 & 0 & f^a \\ 0 & \delta_{a\,b} & 0 & v^a \\ v^a & -f^a & 1 & r \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $a, b = 1, \ .. n$. These may be integrated to equations of the form

$$\tilde{p}^a = \varphi^a(p, q, e, t) = \varphi^a(p, t) = p^a + p^a(t),$$
$$\tilde{q}^b = \varphi^{n+b}(p, q, e, t) = \varphi^{n+b}(q, t) = q^b + q^b(t),$$
$$\tilde{e} = \varphi^{2\,n+1}(p, q, e, t) = e + H(p, q, t),$$
$$\tilde{t} = \varphi^{2\,n+2}(p, q, e, t) = \varphi^{2\,n+2}(t) = t,$$

with the condition that Hamilton's equations are satisfied

$$\frac{d\,q^a(t)}{d\,t} = v^a = \frac{\partial H(p,q,t)}{\partial p^a}, \quad \frac{d\,p^a(t)}{d\,t} = f^a = -\frac{\partial H(p,q,t)}{\partial q^a}, \quad \frac{\partial H(p,q,t)}{\partial t} = r$$

Hamilton relativity group

The final condition to apply is the invariance of length in the inertial rest frame. Setting

$$d\,q^2 \;=\; {}^t d\,z\cdot\eta^q\cdot d\,z = \eta_{i\,j}\,d\,q^i\,d\,q^j$$

Then requiring ${}^t\Phi(\epsilon,\,A,\,0,\,0,\,0)\cdot\eta^q\cdot\Phi(\epsilon,\,A,\,0,\,0,\,0)=\eta^q$ together with the symplectic condition ${}^tA\cdot\zeta^\circ\cdot A=\zeta^\circ$ results in

$$A=\begin{pmatrix} R & 0\\ 0 & R\end{pmatrix}\text{ with } R\in\mathcal{O}(n)$$

Consequently the Hamilton group is

$$\Phi(R,\;f,\,v,\,r)\in\hat{\mathcal{H}}\,a(n)\simeq\mathcal{D}_2\otimes_s\mathcal{O}(n)\otimes_s\mathcal{H}(n)\simeq\mathcal{D}_4\otimes_s\mathcal{SO}(n)\otimes_s\mathcal{H}(n)$$

The Euclidean group is the inertial special case of the Hamilton group

$$\Lambda(\epsilon,\,R,\,v)\simeq\Phi(\epsilon,\,R,\,0,\,v,\,0)\in\hat{\mathcal{E}}(n)\simeq\mathcal{D}_2\otimes_s\mathcal{O}(n)\otimes_s\mathcal{T}(n)\simeq\mathcal{D}_4\otimes_s\mathcal{SO}(n)\otimes_s\mathcal{T}(n)$$

$\hat{\mathcal{H}}\,a(n)$ is the group of transformations between noninertial frames in classical mechanics

It will be convenient in what follows to consider a different ordering of the basis: $d\,z=(d\,t,\;d\,q,\;d\;p,\;d\,e)$ in which

$$\Phi(\epsilon,\,R,\;f,\,v,\,r)=\begin{pmatrix}\epsilon & 0 & 0 & 0\\ v & R & 0 & 0\\ f & 0 & R & 0\\ r & -f & v & \epsilon\end{pmatrix},\text{ in 1 dimension }\;\Phi(f,\,v,\,r)=\begin{pmatrix}1 & 0 & 0 & 0\\ v & 1 & 0 & 0\\ f & 0 & 1 & 0\\ r & -f & v & 1\end{pmatrix}$$

First summary

Euclidean group is relativity group for inertial frames: $\Lambda(\epsilon, R, v) \in \hat{\mathcal{E}}(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{SO}(n) \otimes \mathcal{T}(n)$. Free particle straight lines are diffeomorphisms of the Euclidean group.

Hamilton group is relativity group for noninertial frames:
$\Phi(\epsilon, R, f, v, r) \in \hat{\mathcal{H}}a(n) \simeq \mathcal{D}_4 \otimes_s \mathcal{SO}(n) \otimes \mathcal{H}(n)$. Hamilton's equations are the diffeomorphisms of the Hamilton group

Euclidean group is a special case of the Hamilton group: $\Lambda(\epsilon, R, v) \simeq \Phi(\epsilon, R, 0, v, 0)$,
$\hat{\mathcal{E}}(n) \subset \hat{\mathcal{H}}a(n)$

Time is invariant: all observers agree on the time subspace spanned by $d t$

There is an absolute inertial rest frame that all observers agree on

Hamilton group is 'as fundamental' as Euclidean group: relativity group for general Hamiltonian particles with noninertial frames: $d p / d t \neq 0$

2. Relativistic case

Special relativity eliminates invariant time and absolute rest frame: absolute inertial frame is still implicit

Lorentz group defining relativity of inertial frames leaves invariant Minkowski line element:
$d s^2 = d t^2 - \frac{1}{c^2} d q^2$. Lorentz group contracts to Euclidean group in limit $c \to \infty$.

What is noninertial counterpart of the Lorentz group that contracts in limit to Hamilton group?

Special relativity

Consider time, position, energy, momentum space $z \in \mathbb{P} \simeq \mathbb{R}^{2n+2}$, $z = (x, y) = (t, q, e, p)$, $p, q \in \mathbb{R}^n$, $t, e \in \mathbb{R}$, $n = 3$ the physical case. Co-tangent space $T^*{}_z\mathbb{P}$ spanned by forms $dz = (dt, dq, de, dp)$. $\Gamma^\circ \in \mathcal{GL}(2n+2)$

Special relativity line element is $ds^2 = dt^2 - \frac{1}{c^2}\, dq^2 = {}^t dx \cdot \eta \cdot dx = {}^t dz \cdot \eta^x \cdot dz$. Momentum energy line element in differential form is $d\mu^2 = dp^2 - \frac{1}{c^2}\, de^2 = {}^t dy \cdot \eta \cdot dy = {}^t dz \cdot \eta^y \cdot dz$.

Invariant symplectic metric is ${}^t dz \cdot \zeta \cdot dz = -de \wedge dt + \delta_{i,j}\, dp^i \wedge dq^j$

Group elements Γ° leaving these line elements and metric invariant are the $(2n+2)\times(2n+2)$ dimensional matrices ${}^t\Gamma^\circ \cdot \zeta \cdot \Gamma^\circ = \zeta$, ${}^t\Gamma^\circ \cdot \eta^x \cdot \Gamma^\circ = \eta^x$, ${}^t\Gamma^\circ \cdot \eta^y \cdot \Gamma^\circ = \eta^y$

$$\zeta = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \eta^x = \begin{pmatrix} \eta & 0 \\ 0 & 0 \end{pmatrix}, \eta^y = \begin{pmatrix} 0 & 0 \\ 0 & \eta \end{pmatrix} \text{ implies } \Gamma^\circ = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}, \quad \Lambda \in \mathcal{O}(1, n)$$

Therefore $\Gamma^\circ \in \mathcal{O}(1, n)$ that is the expected group for inertial frames.

For $n = 1$ and ordering basis (dt, dq, dp, de) the explicit group elements are

$$\Gamma^\circ(v) = \gamma^\circ(v) \begin{pmatrix} 1 & v/c^2 & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & v/c^2 \\ 0 & 0 & v & 1 \end{pmatrix}, \qquad \gamma^\circ(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

$$\Gamma^\circ(v'') \cdot \Gamma^\circ(v') = \Gamma^\circ(v) = \Gamma^\circ\!\left(\frac{v'' + v'}{1 + v''\, v'/c^2}\right), \Gamma^{-1}(v) = \Gamma(-v)$$

Reciprocal relativity of noninertial frames: Born-Green metric

For non-inertial frames, follow Born conjecture and combine the line elements to define the Born-Green metric. This is a new physical assertion

$$d s^2 = -d t^2 + \frac{1}{c^2} d q^2 + \frac{1}{b^2} (d p^2 - \frac{1}{c^2} d e^2),$$

where b is a new dimensional physical constant (dimensions of force - Newtons). Planck scales can be defined in terms of the three constants $\{c, b, \hbar\}$ rather than usual $\{c, G, \hbar\}$. Note that $G = \alpha_G \frac{c^4}{b}$, $b \approx \alpha_G 10^{44}$ Newtons.

$$\lambda_t = \sqrt{\frac{\hbar}{b c}}, \; \lambda_q = \sqrt{\frac{\hbar c}{b}}, \; \lambda_p = \sqrt{\frac{\hbar b}{c}}, \; \lambda_e = \sqrt{\hbar b c}.$$

Scaling set by $z = (t/\lambda_t, q/\lambda_q, e/\lambda_e, p/\lambda_p)$ and $(v/c, f/b, r/b c)$

If $\alpha_G = 1$ these are numerically the usual Planck scales.

Terms dependent on inverse powers of b will be manifest only in a very strong interacting regime where forces between particle states approach b (just as effects dependent on inverse powers of c manifest only when velocities between particle states approach c)

Reciprocal relativity of noninertial frames: Unitary group

Consider group leaving the Born-Green metric and symplectic metric invariant: ${}^t\Gamma \cdot \zeta \cdot \Gamma = \zeta$, ${}^t\Gamma \cdot \eta^b \cdot \Gamma = \eta^b$. In basis $d\,z = (d\,t,\ d\,q,\ d\,e,\ d\,p)$

$$\zeta = \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}, \qquad \eta^b = \begin{pmatrix} \eta & 0 \\ 0 & \frac{1}{b^2}\,\eta \end{pmatrix},$$

Group leaving Born-Green metric invariant is $\mathcal{O}(2,\ 2\,n)$.

Group leaving symplectic metric invariant is $\mathcal{S}p(2\,n+2)$.

Group leaving both metrics invariant is

$$\mathcal{O}(2,\ 2\,n) \cap \mathcal{S}p(2\,n+2) \simeq \mathcal{U}(1,\ n)$$

Unitary group may be factored: $\mathcal{U}(1,\ n) \simeq \mathcal{D}_4 \otimes_s \mathcal{U}(1) \otimes_s \mathcal{SU}(1,\ n)$

Reciprocal relativity $\mathcal{SU}(1, 1)$ matrix group

Consider the one dimensional case and consider first the $\Gamma(v, f, r) \in \mathcal{SU}(1, 1)$ group realized by matrix with basis ordering $d z = \{d t, d q, d p, d e\}$. Then ${}^t\Gamma \cdot \zeta \cdot \Gamma = \zeta$, ${}^t\Gamma \cdot \eta^b \cdot \Gamma = \eta^b$ and $\det \Gamma = 1$:

$$\zeta = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \eta^b = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{c^2} & 0 & 0 \\ 0 & 0 & \frac{1}{b^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{c^2 b^2} \end{pmatrix}, \Gamma(v, f, r) = \gamma \begin{pmatrix} 1 & \frac{v}{c^2} & \frac{f}{b^2} & -\frac{r}{b^2 c^2} \\ v & 1 & \frac{r}{b^2} & \frac{-f}{b^2} \\ f & -\frac{r}{c^2} & 1 & \frac{v}{c^2} \\ r & -f & v & 1 \end{pmatrix}$$

with $\gamma = (1 - v^2 / c^2 - f^2 / b^2 + r^2 / b^2 c^2)^{-1/2}$.

Limits are:

$$\Gamma^\circ(v) = \Gamma(v, 0, 0) \in \mathcal{SO}(1,1),$$

$$\Gamma^\circ(v) = \gamma^\circ \begin{pmatrix} 1 & \frac{v}{c^2} & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{v}{c^2} \\ 0 & 0 & v & 1 \end{pmatrix},$$

$$\mathcal{SO}(1, n) \subset \mathcal{SU}(1, n),$$

$$\lim_{b,c\to\infty} \Gamma(v, f, r) = \Phi(v, f, r),$$

$$\Phi(v, f, r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ f & 0 & 1 & 0 \\ r & -f & v & 1 \end{pmatrix},$$

$$\lim_{b,c\to\infty} \mathcal{SU}(1, n) = \mathcal{H}a(n)$$

Transformation equations

Transformation equations are $d\,\tilde{z} = \Gamma \cdot d\,z$

$$d\,\tilde{t} = \gamma\left(d\,t + \frac{v}{c^2}\,d\,q + \frac{f}{b^2}\,d\,p - \frac{r}{b^2\,c^2}\,d\,e\right),$$
$$d\,\tilde{q} = \gamma\left(d\,q + v\,d\,t + \frac{r}{b^2}\,d\,p - \frac{f}{b^2}\,d\,e\right),$$
$$d\,\tilde{p} = \gamma(d\,p + f\,d\,t - \frac{r}{c^2}\,d\,q + \frac{v}{c^2}\,d\,e),$$
$$d\,\tilde{e} = \gamma(d\,e + v\,d\,p - f\,d\,q + r\,d\,t).$$

with $\gamma = (1 - v^2/c^2 - f^2/b^2 + r^2/b^2\,c^2)^{-1/2}$.

Note that the dimensional scaling is given by $d\,z = \left(\frac{d\,t}{\lambda_t}, \frac{d\,q}{\lambda_q}, \frac{d\,p}{\lambda_p}, \frac{d\,e}{\lambda_e}\right), \left(\frac{v}{c}, \frac{f}{b}, \frac{r}{c\,b}\right)$

$$\frac{d\,\tilde{t}}{\lambda_t} = \gamma\left(\frac{d\,t}{\lambda_t} + \frac{v}{c}\frac{d\,q}{\lambda_q} + \frac{f}{b}\frac{d\,p}{\lambda_p} - \frac{r}{c\,b}\frac{d\,e}{\lambda_e}\right),$$
$$\frac{d\,\tilde{q}}{\lambda_q} = \gamma\left(\frac{d\,q}{\lambda_q} + \frac{v}{c}\frac{d\,t}{\lambda_t} + \frac{r}{c\,b}\frac{d\,p}{\lambda_p} - \frac{f}{b}\frac{d\,e}{\lambda_e}\right),$$
$$\frac{d\,\tilde{p}}{\lambda_p} = \gamma\left(\frac{d\,p}{\lambda_p} + \frac{f}{b}\frac{d\,t}{\lambda_t} - \frac{r}{c\,b}\frac{d\,q}{\lambda_q} + \frac{v}{c}\frac{d\,e}{\lambda_e}\right),$$
$$\frac{d\,\tilde{e}}{\lambda_e} = \gamma\left(\frac{d\,e}{\lambda_e} + \frac{v}{c}\frac{d\,p}{\lambda_p} - \frac{f}{b}\frac{d\,q}{\lambda_q} + \frac{r}{c\,b}\frac{d\,t}{\lambda_t}\right).$$

Spacetime itself is relative to noninertial frames. Different noninertial observers measure different spacetime subspaces; the energy and momentum *mix* with position and time.

Group composition

Group multiplication is

$$\Gamma(v'', f'', r'') \cdot \Gamma(v', f', r') = \Gamma(v, f, r)$$

where

$$v = \left(v'' + v' + \frac{1}{b^2}(r' f'' - f' r'')\right) \Big/ \left(1 + \frac{v' v''}{c^2} + \frac{f' f''}{b^2} - \frac{r' r''}{b^2 c^2}\right),$$

$$f = \left(f'' + f' + \frac{1}{c^2}(-r' v'' + v' r'')\right) \Big/ \left(1 + \frac{v' v''}{c^2} + \frac{f' f''}{b^2} - \frac{r' r''}{b^2 c^2}\right),$$

$$r = (r'' + r' - f' v'' + v' f'') \Big/ \left(1 + \frac{v' v''}{c^2} + \frac{f' f''}{b^2} - \frac{r' r''}{b^2 c^2}\right)$$

Null surfaces generalize from $\frac{v^2}{c^2} = 1$. One solution with $r = 0$

$$\frac{v^2}{c^2} + \frac{f^2}{b^2} = 1$$

Forces are relative to particle states. Forces are 'bounded by b'.

Velocities are relative to particle states. Velocities are 'bounded by c'.

Theory consistent with this relativity cannot have force singularities (such as experienced by $1/r^2$ dependencies in usual theory)

Spacetime itself is relative to noninertial frames. There is no absolute inertial frame nor an absolute rest frame.

We call this *reciprocal relativity* after Max Born's concept of *reciprocity*.

Limit $b \to \infty$ transformation equations

Limit $b \to \infty$ transformation equations are

$$d\,\tilde{t} = \gamma^{\circ}(d\,t + \tfrac{v}{c^2}\,d\,q),$$
$$d\,\tilde{q} = \gamma^{\circ}(\,d\,q + v\,d\,t),$$
$$d\,\tilde{p} = \gamma^{\circ}(d\,p + f\,d\,t - \tfrac{r}{c^2}\,d\,q + \tfrac{v}{c^2}\,d\,e),$$
$$d\,\tilde{e} = \gamma^{\circ}(d\,e + v\,d\,p - f\,d\,q + r\,d\,t).$$

with $\gamma^{\circ}(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ and group composition is

$$v = (v' + v'')/(1 + \tfrac{v'\,v''}{c^2}),$$
$$f = (f'' + f' + \tfrac{1}{c^2}(v'\,r'' - r'\,v''))/(1 + \tfrac{v'\,v''}{c^2}),$$
$$r = (r'' + r' - f'\,v'' + v'\,f'')/(1 + \tfrac{v'\,v''}{c^2}).$$

as expected.

Line element contracts as.

$$d\,s^2 = -d\,t^2 + \tfrac{1}{c^2}\,d\,q^2 + \tfrac{1}{b^2}\,d\,p^2 - \tfrac{1}{b^2\,c^2}\,d\,e^2 \underset{b\to\infty}{\to} -d\,t^2 + \tfrac{1}{c^2}\,d\,q^2 \underset{c\to\infty}{\to} -d\,t^2$$

Reciprocal relativity $\mathcal{D}_4 \otimes \mathcal{U}(1)$

The elements of $\Delta \in \mathcal{D}_4$ are again a 4 element discrete group generated by parity and time-energy reversal. In the basis $(d\,t, d\,q, d\,p, d\,e)$, matrices are $(\epsilon_1 = \pm 1,\ \epsilon_2 = \pm 1)$.

$$\Delta = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_2 & 0 & 0 \\ 0 & 0 & \epsilon_2 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}$$

$\Theta \in \mathcal{U}(1)$ are given by

$$\Theta = \begin{pmatrix} \cos\theta & 0 & 0 & -\frac{1}{c\,b}\sin\theta \\ 0 & \cos\theta & -\frac{c}{b}\sin\theta & 0 \\ 0 & \frac{b}{c}\sin\theta & \cos\theta & 0 \\ c\,b\sin\theta & 0 & 0 & \cos\theta \end{pmatrix}$$

For limiting behavior, set $r^\circ = c\,b\tan\theta$. Then $\sin\theta = \frac{r^\circ}{b\,c\sqrt{1+(\frac{r^\circ}{b\,c})^2}}$, $\cos\theta = \frac{1}{\sqrt{1+(\frac{r^\circ}{b\,c})^2}}$

$$\lim_{b\to\infty}\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{r^\circ}{c^2} & 1 & 0 \\ r^\circ & 0 & 0 & 1 \end{pmatrix}, \quad \lim_{b,\,c\to\infty}\Theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ r^\circ & 0 & 0 & 1 \end{pmatrix},$$

Second summary

New physical hypothesis is the Born-Green metric .

Reciprocal relativity of noninertial frames given by $\mathcal{U}(1, n)$ transformations.

No absolute rest frame, no absolute inertial frame.

Spacetime itself is relative to noninertial observer frame, generalized dilation and contractions transform energy-momentum into position-time and vice versa.

Velocities are between particle states. Forces are between particle states.

Velocity and force are bounded. Natural dimensional basis is $\{c, b, \hbar\}$.

In $b \to \infty$ limit spacetime is again invariant. A global inertial frame emerges. It appears that forces are relative to this global inertial frame.

(This is analogous to the special relativity case where, in the limit $c \to \infty$, it appears that velocities are relative to a global inertial rest frame.)

Reciprocally relativistic quantum mechanics

The Unitary group $\mathcal{U}(1, n)$ defines transformations between noninertial frames of reciprocal relativity with invariant symplectic metric and Born-Green line element.

The nonrelativistic limit (group contraction $b, c \to \infty$) is the Hamilton group with invariant Newtonian time line element and invariant symplectic metric

Quantum states are represented as rays in a Hilbert space

Reciprocally relativistic quantum mechanics is understood in terms of the projective representations of the inhomogeneous $\mathcal{U}(1, n)$ group .

Projective representations of a group are equivalent to the unitary representations of its central extension. The central extension may be algebraic and/or topological (cover). For inhomogeneous unitary group, the central extension is the cover of the quaplectic group $\mathcal{Q}(1, n)$.

$$\mathcal{Q}(1, n) \simeq \mathcal{U}(1, n) \otimes_s \mathcal{H}(n+1), \qquad \overline{\mathcal{Q}}(1, n) \simeq \mathcal{D}_4 \otimes_s \mathcal{T}(1) \otimes_s \mathcal{SU}(1, n) \otimes_s \mathcal{H}(n+1)$$

The representation of the algebra of $\mathcal{H}(n+1)$ is the 'p, q' , 'e, t' Heisenberg commutation relations.

Single particle wave equations arise from the Hermitian representations of the Casimir invariants in the enveloping algebra of the quaplectic group. The 'scalar case' is the relativistic oscillator. In general, the wave equations are second order equations of wavefunctions $\psi(q, t)$ that appear to be 'towers of spinning oscillators'.

Quaplectic group

Element of quaplectic group is $g(\Gamma, w, \iota) \in \mathcal{Q}(1, n) = \mathcal{U}(1, n) \otimes \mathcal{H}(n+1)$,

$$g\,(\Gamma, w, \iota) \simeq \begin{pmatrix} \Gamma & 0 & w \\ -\epsilon\;{}^{t}w\cdot\zeta^{\circ}\cdot\Gamma & 1 & \iota \\ 0 & 0 & 1 \end{pmatrix},$$

$$w = \{x, y\},\;\; x = \{t, q\},\;\; y = \{e, t\},\;\; z = \frac{1}{\sqrt{2}}\,(x + i\,y),\;\; \Gamma = \begin{pmatrix} \Lambda & M \\ M & \Lambda \end{pmatrix},\;\; \Upsilon = \frac{1}{2}\,(M + i\,\Lambda),$$

$$g(\Upsilon, z, \iota) \simeq \begin{pmatrix} \Upsilon & 0 & z \\ \overline{\Upsilon\cdot z} & 1 & \iota \\ 0 & 0 & 1 \end{pmatrix}.$$

Lie algebra is $Z_a^{\pm}\; \frac{1}{\sqrt{2}}\,(X_a \pm i\,Y_a)$, $A_{a\,b} = \frac{1}{2}\,(M_{a\,b} + i\,L_{a\,b})$, $a, b = 0, 1, \;..\, n$

$$[A_{a\,b}, A_{c\,d}] = i(\eta_{a\,d}\,A_{c\,b} - \eta_{b\,c}\,A_{a\,d}),\quad [Z_a^+, Z_b^-] = i\,\eta_{a\,b}\,I,$$
$$[A_{a\,b}, Z_c^+] = -i\,\eta_{a\,c}\,Z_b^+,\qquad [A_{a\,b}, Z_c^-] = i\,\eta_{b\,c}\,Z_a^-,$$

Casimir invariants

$$C_0 = I,\qquad C_1 = \eta^{a\,b}\,W_{a\,b}\,,\qquad \ldots$$
$$C_4 = \eta^{a\,b}\,\eta^{c\,d}\,\eta^{e\,f}\,\eta^{g\,h}\,W_{h\,a}\,W_{b\,c}\,W_{d\,e}\,W_{f\,g},$$

where $W_{a\,b} \doteq Z_a^+\,Z_b^- - I\,A_{a\,b}$.

Quantum mechanics: Projective representations of the group

The projective representations of the inhomogeneous Lorentz group define the single particle state space of the inertial case. These are equivalent to the unitary irreducible representations ϱ of the Poincaré group . The particle wave equations are the solution of the eigenvalue equations for the representations of the Casimir operators

$$\varrho'(C_\alpha)|\psi\rangle = c_\alpha |\psi\rangle: \varrho'(Y_a Y^a)|\psi\rangle = \mu^2 |\psi\rangle,$$

$$\varrho'(Y_a Y^a)|\psi\rangle = \mu^2 |\psi\rangle, \varrho'(W_a W^b)|\psi\rangle = s(s+1)\mu^2 |\psi\rangle, W_a = \epsilon_a^{b\,c\,d} L_{b\,c} Y_d$$

The projective representations of the inhomogeneous Unitary group define the single particle state space of the noninertial case. These are equivalent to the unitary irreducible representations ϱ of the cover of the Quaplectic group . The particle wave equations may be derived from the solution of the eigenvalue equations for the Casimir operators

$$\varrho'(C_\alpha)|\psi\rangle = c_\alpha |\psi\rangle$$

$$\varrho'(I)|\psi\rangle = c_0 |\psi\rangle, \quad \varrho'(X_a X^a + Y_a Y^a - I\,U)|\psi\rangle = c_1 |\psi\rangle,$$
$$\eta^{a\,d}\eta^{b\,c}\varrho'(Z_a^+ Z_b^- - I\,A_{a\,b})(Z_c^+ Z_d^- - I\,A_{c\,d})|\psi\rangle = c_2 |\psi\rangle, \ldots.$$

The Weyl-Heisenberg, Poincaré and quaplectic group are semidirect product groups for which the unitary representations may be determined by Mackey's theorems

One slide sketch of Mackey representations

For a semi-direct product group $\mathcal{G} = \mathcal{K} \otimes_s \mathcal{N}$ * where $\mathcal{N}$ is the normal subgroup and $\mathcal{K}$ is the homogeneous group and assume we know the UIR $(\mathcal{K}, \sigma, \mathrm{H}^{\sigma})$ and $(\mathcal{N}, \xi, \mathrm{H}^{\xi})$ then the UIR ϱ of $\mathcal{G}$ acting on H^{ϱ} are determined by the following method.

Determine the *little group(s)* $\mathcal{K}^{\circ}$ through fixed point of natural action $k[\xi] = [\xi]$ of elements $k \in \mathcal{K}$ on the unitary dual $\hat{\mathcal{N}}$ (UIR of $\mathcal{N}$) given by $(k\,\xi)(n) = \rho(k)\,\xi(n)\,\rho(k)^{-1}\ \forall\ n \in \mathcal{N}$. Stabilizers are $\mathcal{G}^{\circ} = \mathcal{K}^{\circ} \otimes_s \mathcal{N}$.

ρ is a projective representation of the stabilizer $\mathcal{G}^{\circ}$ on H^{ξ} that is an extension of ξ, $\rho(n) = \xi(n)\ \forall\ n \in \mathcal{N}$. If $\mathcal{N}$ is abelian, the extension is trivial, $\rho(k) = 1\ \forall\ k \in \mathcal{K}$

If $\mathcal{N}$ is abelian, $\mathrm{H}^{\xi} \simeq \mathbb{C}$, construct representation $\varrho^{\circ} = \sigma \otimes \xi$ acting on Hilbert space $\mathrm{H}^{\sigma} \otimes \mathbb{C}$.

If $\mathcal{N}$ is not abelian, construct representation $\varrho^{\circ} = \sigma \otimes \rho$ acting on Hilbert space $\mathrm{H}^{\sigma} \otimes \mathrm{H}^{\xi}$.

If $\mathcal{G}^{\circ} = \mathcal{G}$ you are done, otherwise need to induce representation on $\mathcal{G}^{\circ}$ to the full group $\mathcal{G}$ acting on the Hilbert space $\mathrm{H}^{\varrho} \simeq \mathrm{H}^{\sigma} \otimes \mathfrak{L}^2(\mathbb{A}, \mathrm{H}^{\xi})$, $a \in \mathbb{A} = \mathcal{G}/\mathcal{G}^{\circ} \simeq \mathcal{K}/\mathcal{K}^{\circ}$. For $\psi \in \mathfrak{L}^2(\mathbb{A}, \mathrm{H}^{\xi})$,

$$(\varrho(g)\,\psi)(a) = \varrho^{\circ}(\Theta(a)^{-1} \cdot g \cdot \Theta(g^{-1}\,a))\,\psi(g^{-1}\,a).$$

* The groups must satisfy certain general technical conditions. A sufficient condition is that the groups are matrix groups that are algebraic subgroups of $\mathcal{GL}(n, \mathbb{R})$

Unitary representations of quaplectic group sketch

Heisenberg group is semidirect product $\mathcal{H}(n) = \mathcal{T}(n) \otimes \mathcal{T}(n+1)$,

$$[Z_a^+, Z_b^-] = i\,\eta_{a,b}\, I.$$

If $\varrho'(I)\,|\psi\rangle = 0$, ($I$ is central element) degenerate abelian case, *Little* group is $\mathcal{T}(n)$.

If $\varrho'(I)\,|\psi\rangle \neq 0$, *Little* group is trivial, from Mackey induction, Hilbert space is $\mathbf{L}^2(\mathbb{R}^n, \mathbb{C})$.

Hermitian representations of the algebra are

$$\langle x|\, \hat{Z}_b^{\pm}\, |\psi\rangle = \langle x|\, \xi'(Z_b^{\pm})\, |\psi\rangle = \left(x^b \pm \frac{\partial}{\partial x_b}\right) \psi(x).$$

Quaplectic group is semidirect product $\mathcal{Q}(1,3) = \mathcal{U}(1,3) \otimes \mathcal{H}(4)$.

For $\varrho'(I)\,|\psi\rangle \neq 0$, *Little* group is $\mathcal{U}(1,3)$, stabilizer is $\mathcal{Q}(1,3)$ and so induction is not required.

Hilbert space is $\mathbb{V}^{\infty} \otimes L^2(\mathbb{R}^n, \mathbb{C})$,

$$\langle x|\, \hat{Z}_b^{\pm}\, |\psi\rangle = \langle x|\, \varrho'(Z_b^{\pm})\, |\psi\rangle = \langle x|\, \xi'(Z_b^{\pm})\, |\psi\rangle = \left(x^b \pm \frac{\partial}{\partial x_b}\right) \psi(x).$$

Representations are

$$\hat{A}_{a\,b} = \rho'(A_{a,b}) = \hat{Z}_a^{+}\, \hat{Z}_b^{-},\ \ \hat{Z}_b^{\pm} = \rho'(Z_b^{\pm}),\ \ \hat{\varepsilon}_{a,b} = \sigma'(A_{a\,b}),$$

$$\varrho'(A_{a\,b}) = \sigma'(A_{a\,b}) + \rho'(A_{a\,b}) = \hat{\varepsilon}_{a\,b} + \hat{Z}_a^{+}\, \hat{Z}_b^{-}.$$

Noninertial particle state wave equations

Wave equations $\hat{C}_\alpha |\psi\rangle = \varrho' (C_\alpha) |\psi\rangle = c_\alpha |\psi\rangle$ are (Jarvis)

$$\eta_{a\,b}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right) = f_1(c_1, d_1)\,\psi(x),$$

$$\hat{\varepsilon}_{b\,a}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right)\psi(x) = f_2(c_1, c_2, d_1, d_2)\,\psi(x),$$

$$\hat{\varepsilon}_b^c\,\hat{\varepsilon}_{c\,a}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right)\psi(x) = f_3(c_1, \ldots c_3, d_1 \ldots d_3)\,\psi(x),$$

$$\hat{\varepsilon}_b^d\,\hat{\varepsilon}_d^c\,\hat{\varepsilon}_{c\,a}\left(x^a + \frac{\partial}{\partial x_a}\right)\left(x^b - \frac{\partial}{\partial x_b}\right)\psi(x) = f_4(c_1, \ldots c_4, d_1, \ldots d_4)\,\psi(x).$$

f_α are specific polynomials, c_1 are Casimir eigenvalues, d_1 are $\mathcal{U}(1, 3)$ Casimir state labels.

It remains to determine whether these reduce to the usual wave equations of special relativistic quantum mechanics in the inertial limit $b \to \infty$.

$$\begin{array}{llllll}
\text{Quaplectic} & -\varrho \to & \text{UIR} & -\text{algebra} \to & A_{a\,b}, Z_a^{\pm}, I & \text{Quaplectic Casimirs} \\
\downarrow b \to \infty & & \downarrow b \to \infty & & \downarrow b \to \infty & \downarrow b \to \infty \\
\text{Poincare} + & -\varrho \to & \text{UIR} & -\text{algebra} \to & L_{a\,b}, Y_a & \text{Poincaré Casimirs} \\
\downarrow c \to \infty & & \downarrow c \to \infty & & \downarrow c \to \infty & \downarrow c \to \infty \\
\text{Hamilton} & -\varrho \to & \text{UIR} & -\text{algebra} \to & J_i, P_i, E, \ldots. & \text{Hamilton Casimirs}
\end{array}$$

Final summary

The Hamilton group generalizes the Euclidean group to general particle motion: noninertial frames

Reciprocal relativity of noninertial frames is given by $\mathcal{U}(1, 3)$ transformations.

New physical assumption is Born-Green metric.

No absolute rest frame, no absolute inertial frame.

Spacetime itself is relative to noninertial observer frame. Generalized dilation and contractions transform energy-momentum into position-time and vice versa.

Velocity is bounded by c and force by b. Natural dimensional basis is $\{c, b, \hbar\}$.

A reciprocal relativistic quantum theory is the projective representations of the inhomogeneous unitary group.

Central extension is the cover of the quaplectic group $\mathcal{U}(1, 3) \otimes_s \mathcal{H}(4)$.

The usual Heisenberg commutation relations therefore arise directly from the requirement of projective representations that result from states being identified as rays in a Hilbert space.

The wave functions are on $\mathbf{L}^2(\mathbb{R}^4, \mathbb{C})$, parameters of any computing set of 4 generators, $\begin{matrix} T & Q_i \\ P_i & E \end{matrix}$

The wave equations may be computed, coupled set of *relativistic spinning oscillators*.