Classical and Quantum Dynamics in the Driven Triangle Well

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- Classical Motion
- Quantum Dynamics
- Radiation Spectra
- Quasienergies
- Periodic Orbit Theory

Classical Motion of the Driven Triangle Well

- Hamiltonian

$$H = \frac{p^2}{2m} - \epsilon_0 x - \epsilon x \cos(\omega t + \theta) + V_L(x)$$

$$V_L(x) = \begin{cases} 0 & \text{if } x < L \\ \infty & \text{if } x > L \end{cases}$$

- Equations of Motion

$$\dot{x} = \frac{p}{m}$$

$$\dot{p} = \epsilon_0 + \epsilon \cos(\omega t + \theta)$$

- Infinite Wall
  - To numerically integrate the equations of motion one must stop the integration when $x = L$ and reverse the momentum.
  - The NAG library has an ode integrator 'd02cje' which stops when $x = L$.

- Collision Maps
  - One way to study the dynamics of this system is to make a plot of the canonical action (J) and phase of the driving field (ϕ) each time the particle hits the infinite wall.
  - These plots give a good indication of what the classical motion is like for this sytem.
  - As we increase ϵ, these plots show the transition to chaotic motion.

Quantum Dynamics of the Driven Triangle Well

- Time-Dependent Schrödinger's Equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = [H_0 - \xi(t)x\cos(\omega t+\theta)]|\psi(t)\rangle$$

$$H_0 = \frac{p^2}{2m} - \epsilon_0 x$$

$$\xi(t) = \begin{cases} \epsilon\sin^2(\frac{\omega t}{4n}) & t < \frac{2\pi n}{\omega} \\ \epsilon & t > \frac{2\pi n}{\omega} \end{cases}$$

- Solution for Unperturbed Triangle Well

$$\psi_i(x) = \langle x|E_i\rangle = N_i\mathrm{Ai}\left[-(\tfrac{2m}{\hbar^2})^{\frac{1}{3}}\epsilon_0^{-\frac{2}{3}}E_i - (\tfrac{2m\epsilon_0}{\hbar^2})^{\frac{1}{3}}x\right]$$

- Decomposition of Time-Dependent Wavefunction

$$|\psi(t)\rangle = \sum_i c_i(t)|E_i\rangle$$

  - Why not solve the PDE?
  - This method makes calculating spectra easy and allows study of the diffusion of energy.

- ODEs for C(t)'s

$$\frac{\partial c_i(t)}{\partial t} = -\frac{iE_i}{\hbar}c_i(t) + \frac{i}{\hbar}\xi(t)\cos(\omega t + \theta)\sum_j x_{ij}c_j(t)$$

$$x_{ij} = \begin{cases} -\frac{2E_i}{3\epsilon_0} & i = j \\ -\frac{\epsilon_0}{m(E_i - E_j)^2} & i \neq j \end{cases}$$

- Acceleration Time Series

$$\langle\psi(t)|\ddot{x}|\psi(t)\rangle = \sum_{ij} c_j^*(t)\ddot{x}_{ij}c_i(t)$$

$$\ddot{x}_{ij} = -\frac{1}{\hbar^2}(E_i - E_j)^2 x_{ij} + \frac{1}{\hbar^2}\xi(t)\cos(\omega t + \theta)\sum_k (2E_k - E_j - E_i)x_{ik}x_{kj}$$

- Radiation Spectra - High Harmonic Generation

Quasienergies

- Floquet Operator

$$\hat{U}_T|\psi(t)\rangle = |\psi(t+T)\rangle$$

- If the system is periodic in time then $\hat{U}_T$ can be diagonalized. The eigenvalues of this operator take the form $e^{-\frac{iq_nT}{\hbar}}$. The q_n are called the quasienergies of the system. They are only defined modulo $\omega\hbar$.

- Quasienergies in time-periodic systems play a role similar to that of energies in conservative systems.

- All of the analysis that might usually be done on the energy spectrum of a conservative system can be done on the quasienergy spectrum of a time-periodic system.

Periodic Orbits and Quasienergies

- Periodic Orbits and Energy Spectra
  - Chaotic Systems - Gutzwiller
  - Regular Systems - Balian and Bloch, Berry and Tabor

- Periodic Orbits and Quasienergy Spectra
  - A theory similar to theperiodic-orbit theory for conservative systems was developed by Tabor to study area-preserving maps.
  - Quasienergy density of states

$$\rho(Q) = \frac{T}{2\pi i\hbar} \sum_{N=-\infty}^{\infty} e^{i(NQT/\hbar)} \mathrm{Tr}(\hat{U}^N) = \sum_{M=-\infty}^{\infty} \sum_n \delta(M\omega\hbar - (Q - q_n))$$

  - $\mathrm{Tr}(\hat{U}^N)$ is a sum of terms from each orbit that is periodic with period NT.

- To construct the quasienergy density of states:

1. locate all periodic orbits of period NT

2. calculate the stability matrix M_j for each orbit

$$\mathbf{M_j} = \prod_{i=0}^{N} \begin{bmatrix} \frac{dp_{i+1}}{dp_i} & \frac{dp_{i+1}}{dx_i} \\ \frac{dx_{i+1}}{dp_i} & \frac{dx_{i+1}}{dx_i} \end{bmatrix}$$

3. calculate residue R_j for each orbit

$$R_j = \frac{1}{4}(2 - \mathrm{Tr}\mathbf{M_j})$$

4. calculate the classical action along each orbit

$$\bar{W}_j = \int_0^{NT} L(t)dt$$

5. repeat above procedure for all N

6. quasienergy density of states

$$\rho(Q) = \frac{T}{2\pi\hbar} \sum_j \frac{1}{2} \left[\frac{1}{R_j}\right]^{1/2} e^{(i/\hbar)(N_j QT + \bar{W}_j)}$$

- Results for driven harmonic oscillator
  - Use shooting technique with Newton's method to find the sole periodic orbit with period T.
  - Calculate residue and action for this orbit and its repetitions.
  - Compare the density of states obtained from this numerical oribt-finding to the analytical result.

- Application to the Driven Triangle Well
  - Infinite wall adds difficulty to ODE integration.
  - The map $t \rightarrow t + T$ is non-linear in the initial coordinates, so several Newton iterations will be required to locate the periodic orbit.
  - There may be more than one periodic orbit with a given period.
  - The basin of attraction of a given periodic orbit is finite.
  - These last three probelems can be handled by using a mesh of starting points from which one searches for periodic orbits.