Killing Vector Fields

Scott H. Hawley
M382G
Dr. Speight
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1 Flows, Isometries and Lie Derivatives

Recall that a smooth vector field X on a manifold M can be used to define a *flow*; that is, a map $c : I \to M$ such that the tangent vector of c at a point $c(t)$ is given by $X(c(t))$, i.e.

$$\frac{dc^\mu(t)}{dt} = X^\mu(c(t)). \tag{1}$$

A vector field which defines a flow in this fashion is said to *generate* the flow. Recall also that this flow implies a local one-parameter family of diffeomorphisms $\phi_t : M \to M$, and that these diffeomorphisms are called *isometries* if they leave the metric unchanged, *i.e.* if

$$\phi_t^* g_{\mu\nu} = g_{\mu\nu}. \tag{2}$$

These diffeomorphisms allow one to pull back tensor fields along the flow, and thus define a notion of a directional derivative called the Lie derivative:
Definition Given a local one-parameter family of diffeomorphisms ϕ_t along a flow $c(t)$ generated by a vector field X, the *Lie derivative* of a general (p, q) tensor T at a point p is given by

$$\mathcal{L}_X T = \lim_{t\to 0} \frac{1}{t}(\phi_{-t}^* T|_{\phi_t(p)} - T|_p).\Box \tag{3}$$

Given two differentiable tensor fields T and U, the Lie derivative in the direction of a vector field X satisfies the following algebraic poperties:

(a) $\mathbb{R}$ multilinearity: $\mathcal{L}_X(aT + bU) = a\mathcal{L}_X T + b\mathcal{L}_X U, (a, b) \in \mathbb{R}$.
(b) Liebniz rule: $\mathcal{L}_X(T \otimes U) = (\mathcal{L}_X T) \otimes U + T \otimes \mathcal{L}_X U$.

From the definition of the Lie derivative (3) the following expressions for functions, vectors and one-forms can be found [1]:

$$\mathcal{L}_X f = X[f] \tag{4}$$

$$\mathcal{L}_X \frac{\partial}{\partial x^i} = -\frac{\partial X^j}{\partial x^i}\frac{\partial}{\partial x^j} \tag{5}$$

$$\mathcal{L}_X dx^i = \frac{\partial X^i}{\partial x^j} dx^j \tag{6}$$

By use of the Leibniz property and the preceeding expressions, one can define the Lie derivative for general tensors. For example, for a (0,2) tensor $T = T_{ij} dx^i \otimes dx^j$,

$$\begin{aligned} \mathcal{L}_X T &= (\mathcal{L}_X T_{ij}) dx^i \otimes dx^j + T_{ij}(\mathcal{L}_X dx^i) \otimes dx^j + T_{ij} dx^i \otimes \mathcal{L}_X dx^j \\ &= X[T_{ij}] dx^i \otimes dx^j + T_{ij}\frac{\partial X^i}{\partial x^k} dx^k \otimes dx^j + T_{ij} dx^i \otimes \frac{\partial X^j}{\partial x^k} dx^k \end{aligned} \tag{7}$$

We see from the definition of the Lie derivative (3), that if the local diffeomorphisms ϕ_t associated with a vector field X are isometries, then the Lie derivative of the Riemannian metric g will vanish. In fact, the necessary and sufficient condition for ϕ_t to be a local group of isometries is $\mathcal{L}_X g = 0$ [6]. This brings us to the discussion of Killing fields.

2 Killing Fields

Definition A *Killing vector field* (or simply a *Killing field*) is a vector field which generates an isometry.□
As noted above, an equivalent definition of a Killing vector field is a vector field ξ such that $\mathcal{L}_\xi g = 0$.
It can be shown that, for a $(0, q)$ tensor T and vector fields $(X, X_1, X_2, ...X_q)$ on a Riemannian manifold M,

$$\begin{aligned} (\mathcal{L}_X T)(X_1, ..., X_q) &= (\nabla_X T)(X_1, ..., X_q) \\ &\quad + \sum_{i=1}^{q} T(X_1, ..., X_{i-1}, \nabla_{X_i} X, ..., X_q) \end{aligned} \tag{8}$$

[2]. Applying this to the metric using the Killing vector field ξ produces *Killing's equation*:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{9}$$

Since the metric does not change along flows generated by Killing fields, the fields describe the *symmetry* of the manifold.

Properties of Killing Fields

Some important properties of Killing fields include:

(i) For two Killing fields ξ and η, the linear combination $aX + bY, (a, b) \in \mathbb{R}$, is also a Killing field.

(ii) The Lie bracket $[\xi, \eta] \equiv \xi\eta - \eta\xi$ of two Killing fields ξ and η is also a Killing field.

(iii) Given a Killing field ξ and geodesic γ with velocity vector u, the quantity $\xi_\mu u^\mu$ is constant along γ.

Proof:

(i) follows from $\mathbb{R}$-multilinearity of the connection.

(ii) A useful identity that follows naturally from the properties of the Lie derivative is

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X.$$

Thus for two Killing fields ξ and η, the Lie derivative of the metric with respect to their Lie bracket is

$$\begin{aligned} \mathcal{L}_{[\xi,\eta]} g &= \mathcal{L}_\xi \mathcal{L}_\eta g - \mathcal{L}_\eta \mathcal{L}_\xi g \\ &= \mathcal{L}_\xi(0) - \mathcal{L}_\eta(0) = 0. \end{aligned} \tag{10}$$

(iii) The change in $\xi_\mu u^\mu$ along γ is given by

$$\begin{aligned} u^\nu \nabla_\nu (\xi_\mu u^\mu) &= u^\nu u^\mu \nabla_\nu \xi_\mu + u^\nu \xi_\mu \nabla_\nu u^\mu \\ &= 0 \end{aligned} \tag{11}$$

The first term is zero by Killing's equation and the second by the geodesic equation.$\square$

One would think that use of property (iii) would produce many interesting physical examples of conservation laws derived from Killing fields. I was unable to find any, aside from the obvious example of conservation of energy in a stationary spacetime. The following is a more involved example with interesting physical results, although few of the properties of Killing fields are exploited.

3 Application: Frame Dragging Near a Kerr-Newman Black Hole [4],[6]

Consider the Kerr-Newman metric for the spacetime geometry near a rotating black hole with angular momentum S, electric charge Q and mass M:

$$ds^2 = -\frac{\Delta}{\rho}[dt - a\sin^2\theta d\phi]^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - adt]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \quad (12)$$

where

$$\Delta \equiv r^2 - 2Mr + a^2 + Q^2 \quad (13)$$

$$\rho^2 \equiv r^2 + a^2\cos^2\theta \quad (14)$$

$$a \equiv S/M. \quad (15)$$

Notice that the metric is independent of the coordinates t and ϕ. This means that there will be two Killing vector fields associated with the stationarity (invariance with respect to time translation) and axial symmetry, repectively:

$$\xi_{(t)} \equiv (\partial/\partial t) \qquad \xi_{(\phi)} \equiv (\partial/\partial\phi). \quad (16)$$

An object orbiting the black hole at constant (r,θ) with uniform angular velocity Ω will thus see no change in the geometry of spacetime in his vicinity. Such an observer is said to be *stationary.* An object moving at constant (r,θ,ϕ) is said to be "static" with repsect to the distant stars. Static observers have velocity vectors which are parallel to the timelike Killing vector field, whose norm is

$$\xi^{\mu}_{(t)}\xi_{(t)\,\mu} = g_{tt} = \frac{a^2\sin^2\theta - \Delta}{\rho^2} = \frac{2Mr - Q^2}{\rho^2} - 1 \quad (17)$$

We see that as r decreases, this norm becomes *positive* in the region where $2Mr - Q^2 - r^2 - a^2\cos^2\theta > 0$. This region extends *outside* the event horizon $(r = 2M - Q^2)$ for $a \neq 0$. In general relativity, objects are constrained to move along timelike (*i.e.* negative norm) geodesics. Thus we see that in the region

$$r < M + (M^2 - Q^2 - a^2 \cos^2 \theta)^{1/2}, \tag{18}$$

called the *ergosphere*, there exist no static observers. In other words, in order for an object to remain in static orbit about the black hole, he would have to orbit faster than the speed of light. Furthermore, if we look at the condition for the curve defined by an observer with velocity vector u,

$$g_{\mu\nu} u^\mu u^\nu < 0, \tag{19}$$

and note that all the terms in this equation are positive except the term $2g_{t\phi} u^t u^\phi$, we get an additional condition for the motion of observers. The four-velocity components are given by

$$u^t = \frac{dt}{d\tau}, \qquad u^\phi = \frac{d\phi}{d\tau,} \tag{20}$$

where τ is the "proper time" *i.e.* the time measured by an observer following the path $c(\tau)$. Since the quantity $dt/d\tau$ is positive and the metric component $g_{t\phi}$ is negative in the ergosphere, we must have, for any particle in the ergosphere,

$$\frac{d\phi}{d\tau} > 0. \tag{21}$$

Thus any particle orbiting within the ergosphere (in stationary but not static orbit) *must orbit in the direction of rotation of the black hole.* This result is known as "dragging of inertial frames" relative to distant stars.

References

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