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P 387N

Project 4

- In this project the classical wave equation is solved by two different implicit methods in one dimension,

$$Q_{,tt} - Q_{,xx} = 0.$$

  The wave velocity is set to unity ($v = 1$). The wave has an initial velocity.

- Periodic boundary conditions were used primarily in this project. Fixed boundary conditions were used on a couple runs. The periodic boundary conditions map the right grid boundary into the left boundary and vice versa.

- The initial data is a gaussian pulse with unit velocity in the positive direction. The analytic solution for this wave is given by the equation

$$Q(t, x) = e^{-200(x-t)^2}$$

  The initial data was set from the above equation with $t = 0$. The initial condition was the time derivative of the analytic solution,

$$\frac{\partial Q}{\partial t} = 400 x e^{-100(x-t)^2}$$

- The first implicit scheme uses a five point numerical molecule. The fundamental FD equation is

$$Q_j^n = 2Q_j^{n-1} - Q_j^{n-2} + \gamma^2(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n)$$

  where $\gamma = v\Delta t/\Delta x$. This scheme is accurate to order $O[(\Delta t), (\Delta x)^2]$. The largest terms in the truncation error

$$\text{TE} = \gamma \Delta x Q_{ttt} + \frac{(\Delta x)^2}{4}(2 - \gamma^2) Q_{xxxx}$$

  The derivative Q_{ttt} can be expressed using the wave equation $Q_{ttt} = Q_{txx}$. This term seems to imply that the largest order error term is dissipative, as is observed. (I say "seems" because strictly one must convert all time derivatives to spatial derivatives for the comparison. However, I was unable to express Q_{xxt} strictly in terms of spatial derivatives using the wave equation.) Additionally, since this term is always positive, the scheme is unconditionally unstable. Notice in the comparative plots that as the grid parameter Δx is increased, that the dissipation increases.

- The second implicit scheme is based on a nine point molecule. The leading order truncation error term is

$$\text{TE} = \frac{1}{6}(\gamma^2 - 1)(\Delta x)^2 Q_{xxxx}$$

Here Q has an even number of spatial derivatives. This implies that the scheme is largely dissipative. Also, if $\gamma = 1$, the error term vanishes—and infact all of the terms in the truncation error vanish, though not shown here—giving the exact solution. There is no Q_{xx} term in the truncation error, so the scheme is unconditionally stable.

- It is constructive to compare the two solutions under a variety of different conditions. First, as mentioned above, the 9-point scheme generates the exact solution when $\gamma = 1$ while the truncation error of the 5-point scheme never completely vanishes. Additionally, the 9-point scheme performs better when γ is close to one, while the 5-point scheme performs best when γ approaches zero. As Δx is increased, making the grid coarser, the magnitude of the truncation error obviously increases. Dissipation dominates the error using the five point scheme, but as the grid is coarsened with the 9-point scheme, dispersion errors also become quite large. These behaviors can be seen in the plots linked to this site.
- The convergence factor was calculated using the standard l_2 norm. (See other information pages for projects 1 or 2 for details.) The convergence factor for the 5-point scheme begins at 2 (as expected for a first order method), but decays exponentially to 1. I suspect that the large amount of dissipation in this scheme slows the convergance. The convergance factor for the 9-point scheme begins at 4, and also decays, but much more slowly. The dissipation in this scheme, while the dominant error term, is an order of magnitude less than in the 5-point scheme.
- The plots were generated by on grids with 100, 200, and 400 points, distributed over the range $[-1, 1]$.