Chapter 4

Stability of Unconstrained ADM Evolutions

4.1 General Comments

The unconstrained evolution of the ADM form of the Einstein equations is the most convenient and efficient numerical approach. We have the fewest number of equations to compute–the six evolution equations–and the Gauss-Codazzi relation assures us that a valid, initial hypersurface will lead to valid, time-advanced hypersurfaces. While this is strictly true for analytical solutions, numeric solutions always allow the possibility that numerical artifacts will be introduced which corrupt the evolved solutions. Thus, numerical techniques are always suspect and the burden lies with the researcher to verify the solutions.

The common difficulty with numerically evolving spacetimes are instabilities. Although first attempts at a numerical scheme may lead to unstable updates, it is usually not too difficult to develop stable interior update schemes, especially for the relatively week field spacetimes. The intransigent instabilities occur at the non-physical, computation boundaries of the spacetime where the standard interior update schemes must be modified for the one-sided grids. For strong fields and for a variety of dynamical coordinate conditions, one must carefully verify the evolutions are proceeding in a stable manner throughout the interior as well as at the grid boundaries.

Recently, arguments have been proposed which question the validity of using unconstrained evolutions for any spacetime. An argument by contradiction has been demonstrated showing that unconstrained evolution of a particular spacetime inherently allows unstable, growing modes. If valid, this argument casts doubt on any unconstrained evolution scheme.

This criticism is addressed in the analysis below with the conclusion that it is not generally valid; unconstrained evolution schemes are not to be summarily dismissed. The constraint equations, although not required as an integral part of the numerical evolution, are still valuable for demonstrating the numerical solution obtained is a valid spacetime. This is necessary since solutions may exist in the Banach space for the algebraic representation of the evolution equation in the numerical scheme which do not correspond to physical spacetimes which satisfy the exact Einstein equations.

4.2 Growing Modes in a Double-Null Coordinate Model Problem

The criticism of unconstrained evolution is presented using a model problem of a spherical spacetime in double-null coordinates. To simplify the analysis, the model problem uses a linearized form of the Einstein equations developed for the spacetime. They then study the solutions of the problem and show unbounded, growing modes are admitted. This is not a problem for solutions of the exact problem if the initial conditions satisfy the constaints. However, in the case of a numerical scheme, truncation error will act as an artificial source and will be allowed to grow in an unbounded manner. The authors demonstrate this using numerical examples.

4.3 Mode Analysis in an ADM Model Problem

The first issue to address regarding this criticism is the appropriateness of comparing the behavior of solutions from linearized equations with solutions from the exact equations. Since the argument will be based on comparing the behavior of these linearized equations with the exact equations, we should restrict ourselves only to solutions in which small excursions are considered since only there will the comparison be valid. If we observe growing modes, we should be wary not to examine the solution space where they become large since the linearized approximation would no longer be valid. For now we will suspend this exception and subject the ADM form of the Einstein equations to a similar mode analysis.

In a like manner, we will take the linearized version of the ADM form to make the analysis tractible. We will then analyze solutions to these linearized equations and look for growing mode behavior. Numerical examples will also be given.

$$G_{ab} = 8\pi T_{ab}, \tag{4.1}$$

4.4 Conclusions

Finally, we revisit the suspended exception regarding unbounded modes in linearized equations and address the applicability of comparing the solutions to the linearized equations with the solutions to the exact equations for both model problems.