Project #9: Eddington-Finkelstein Coordinates

1 The Schwarzschild Metric

We're now ready to calculate numerical solutions of pulses in the vicinity of non-rotating black holes. One could describe this space-time using the Schwarzschild metric.:

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\Omega$$

But this metric has a coordinate singularity at the Schwarzschild radius $R = 2M$. So one could instead make a transformation to Eddington-Finkelstein coordinates, which uses a different t coordinate.:

$$\begin{aligned} t' &= t + 2Mln|r-2M| & r > 2M \\ t' &= t + 2Mln|2M-r| & r < 2M \end{aligned}$$

The metric then has the form.:

$$\begin{aligned} ds^2 &= -(1-\frac{2M}{r})dt^2 + \frac{4M}{r}dtdr + (1+\frac{2M}{r})dr^2 + r^2 d\Omega \\ g^{00} &= -(1+\frac{2M}{r}) \\ g^{01} &= \frac{2M}{r} \\ g^{11} &= 1-\frac{2M}{r} \\ g^{22} &= \frac{1}{r^2} \\ g^{33} &= \frac{1}{r^2 sin\theta} \end{aligned}$$

One can determine the 1-D wave equation for this particular metric using the equation.:

$$0 = \frac{1}{\sqrt{-g}}(\sqrt{-g}Q_{,\alpha}g^{\alpha\beta})_{,\beta}$$

For now, one ignores g^{22} and g^{33} in calculating the determinate g. g =-1 then, and the 1-D wave equation is.

$$0 = -(1+\frac{2M}{r})\frac{\partial^2 Q}{\partial t^2} + \frac{4M}{r}\frac{\partial^2 Q}{\partial r \partial t} + (1-\frac{2M}{r})\frac{\partial^2 Q}{\partial r^2} - \frac{2M}{r^2}\frac{\partial Q}{\partial t} + \frac{2M}{r^2}\frac{\partial Q}{\partial r}$$

which can be factored to:

$$0 = (\frac{\partial}{\partial r} - \frac{\partial}{\partial t})((1-\frac{2M}{r})\frac{\partial}{\partial r} + (1+\frac{2M}{r})\frac{\partial}{\partial t})Q$$

Each of these factors corresponds to a pulse solution, and one uses these factors to set up the radiation boundary conditions.

$$
\begin{aligned}
0 &= (\frac{\partial}{\partial r} - \frac{\partial}{\partial t})g && (1)\\
0 &= ((1 - \frac{2M}{r})\frac{\partial}{\partial r} + (1 + \frac{2M}{r})\frac{\partial}{\partial t})f && (2)
\end{aligned}
$$

$$
\begin{aligned}
g &= g(r+t) \\
f &= f(r + 4Mln|\tfrac{r-2M}{r_0-2M}| - t) && r > 2M \\
f &= f(r + 4Mln|\tfrac{2M-r}{2M-r_0}| - t) && r > 2M
\end{aligned}
$$

where $r0$ is an arbitrary constant.

2 Results

This 1-D equation introduces another partial derivative that one has to expand.: $\partial Q/\partial t$ One uses an expansion similar to that used for $\partial Q/\partial x$.:

$$
\begin{aligned}
\delta_t Q_j^n &= Q_j^{n+1} - Q_j^{n-1} \\
0 &= \frac{\theta_t}{2}(\frac{\delta_x Q_{j+1}^n}{2\Delta t} + \frac{\delta_x Q_{j-1}^n}{2\Delta t}) + (1-\theta_t)\frac{\delta_x Q_j^n}{2\Delta t} + E_t
\end{aligned}
$$

Now one has four different θ parameters, and one names them θ_{rr}, θ_{tt}, θ_r, and θ_t corresponding to their partial derivatives. Also, r can only have positive values, and the coordinates still have an intrinsic singularity at r =0.0. So one solves the wave equation for a range r =[1.0, 5.0], varying the value of the black hole mass M.

The g solution describes a pulse moving left at a speed v = 1.0 that never changes its shape or velocity, even when crossing R. As for the f solution, any part of f lying right on R (provided that f is defined at R) stays at R forever. The parts of f to the right of R flow outward while the parts to the left fall inwards to the intrinsic singularity. Also, both parts expand as they distance themselves from R.

If $R \leq$1.0, g-pulses could enter the domain through the right boundary, and f-pulses could enter through the left. So one applies Equation (1) to the right boundary and (2) to the left to ensure that only solutions that originate in the domain can exist. But if $R \geq$5.0, the domain is entirely within the black hole. All solutions must fall to the singularity, to the left. So one applies both Equations (1) and (2) to the right boundary, leaving no conditions on the left.

The situation becomes complicated when R is within the domain, though. In this case, one still applies (1) to the right boundary to prevent incoming g-pulses. But **both** solutions cross the left boundary. I have found that if I use Equation (2) on the left as with R <1.0 the matrix has a condition number on the order of 10^{15}.

My first solution was to break the matrix in two. If R corresponded to point Nb and the boundaries corresponded to 0 and Nf, I defined $Q1$(-1:1,0:$Nb-1$) for the interior and $Q2$(-1:1,Nb:Nf) for the exterior. $Q2$ I could easily solve the way I solved for Q for $R \leq$1.0. Then for $Q1$ I applied Equation (1) to its right boundary (the Shwarzschild radius), using a molecule centered on Nb . The matrix row correponding this boundary condition would then have only two non-zero components, and I used the values of $Q2$(-1:1,Nb) to define the right side. Similarly, I had a symmetric molecule solution centered at $Nb - 1$, whose row also had only two non-zero compoments and used $Q2$(-1:1,Nb) to build its right side. These two rows enabled me to solve for $Q1$ without imposing any boundary conditions on the left boundary.

But this method required twice as many matrices and arrays as before. So Dave Nielson suggested to me that I simply keep one whole matrix and apply Equation (1) to a molecule **centered at** Nb . In this case I would have two boundary conditions, one at the right boundary and one at the Shwarzschild radius. The Shwarzschild radius boundary condition would allow only g-pulses, which cross the radius without distortion, to exist at the radius. This set up turned out to work flawlessly, and much more economically than the two matrix method. The condition number stayed within a reasonable range of 6 to 70.

2.1 g-pulses

Movies # 1 to # 3 show a g-pulse shaped like a gaussian with a width of 1.2, with ρ =1.0 and all the θ's equal to 0.5. The Shwarzschild radius R =0.5, 3.0, and 6.0 respectively. But all three movies look exactly alike, and resemble a left-going pulse for a 1-D wave equation with β =0. So the radius does not affect this particular solution as predicted.

One has some trouble graphing the convergence factor, though. Figure # 1 shows that the factor for Movie # 2 is only 2.159. But when one sets ρ =0.99 the factor does reach 4.0, as shown in Figure # 3 (although when the pulse starts to clear the left boundary the factor falls). The graphs exhibit behavior similarly seen in Project # 4. Figures # 2 and # 4 show that the factors for $R = 0.5$ and 6.0 also plateau at 4.0 for this value of ρ briefly. But the factor in # 2 eventually increases irregularly, while # 4 decreases.

The value of the R might not be certain. So we vary the value of Nb greater and lesser than the value corresponding to R, for the case of g-pulses. With Nb varied either way, the condition number tends to increase. The case shown in Movie # 2 has a condition number of around 44.4. But this number increases to about 150.0 if Nb is shifted away from R by an amount 0.2. Also, the solution starts to blow up at R when the pulse has nearly cleared the left boundary. Shifting Nb further increases the condition number even more, and the solution blows up earlier.

2.2 f-pulses

As initial conditions for f-pulses, the program also uses gaussians with widths of 1.2. r_0 is chosen such that the pulse has its peak at r =3.0 for $M \leq 0.5$ or $M \geq 2.5$. For M between those values, the program defines an exterior pulse centered a distance of 1.0 outside the black hole and an interior pulse centered 0.5 inside. Movies # 4 - 6 show these pulses, which all obey predictions. The pulses are all zero at R and are compressed due to the ln term in the argument of f. Both kinds of pulses stay on their sides of R all the time, and gradually expand as they get further away from R. Also, the pulses tend to move more slowly than v =1.

Figures # 5 to 7 show the convergence factors for these movies. We do not have the problem with f-pulses that we had in Figure # 1, probably because the f-pulses have to change shape over time. Figure # 5 shows bumps at around time $t = 3.0$, the time when the pulse's peak clears the right boundary. The graph in Figure # 6 hovers at higher than 4.0 until about t = 5.0. The exterior pulse crosses the right boundary at that time, and the convergence factor starts to vary from its smooth shape. Between 6.0 and 7.0 the interior pulse clears the left boundary. The graph has a correpsonding dip in the value of the convergence factor. Also, the solutions with the coarser grids tended to have more small g-pulse errors due the inaccuracy of the gridding. The pulses corresponding to Figure 7 tend to leave a tail of values of Q on the order of 10^{-20} and lower. So once the pulse clears the left boundary at t =5.6, the condition number jumps up considerably.