Project #7: Implicit Scheme Using a Variable Shift Factor

1 Introduction

Now one can work out the situation when the shift factor can vary with respect to the space and time coordinates. For a general expression of β, the one dimensional wave equation takes the form:

$$0 = (\frac{1}{g_{11}} - \beta^2)\frac{\partial^2 Q}{\partial x^2} + 2\beta\frac{\partial^2 Q}{\partial x \partial t} - \Gamma\frac{\partial Q}{\partial x} - \frac{\partial^2 Q}{\partial t^2} \tag{1}$$

g_{11} comes from the metric of the shifted frame of reference, and Γ can be determined from the transformation law between the shifted frame and the frame of reference with $\beta = 0.0$. These new variables may make it necessary for the program to calculate the matrix for each time step, making a run considerably longer than before. Also, one needs to derive the expansion terms corresponding to $\partial_x Q$:

$$\begin{aligned} \delta_x Q_j^n &= Q_{j+1}^n - Q_{j-1}^n \\ 0 &= \frac{\theta_f}{2}(\frac{\delta_x Q_j^{n+1}}{2\Delta x} + \frac{\delta_x Q_j^{n-1}}{2\Delta x}) + (1-\theta_f)\frac{\delta_x Q_j^n}{2\Delta x} + E_x \end{aligned}$$

This introduces a new parameter, θ_f.

For this specific project one uses $\beta = ax$, with a a constant. One can then calculate $g_{11} = exp(2at)$ and $\Gamma = a^2 x$, and the solutions change to $Af(xexp(at) - t)$ and $Bg(xexp(at) + t)$. If one sets the inside of Af equal to a value x_0 one can derive the equation

$$x = (x_0 + t)e^{-at} \tag{2}$$

which describes the path of a specific point on the solution that starts at $x = x_0$ at time $t = 0.0$. Also, the equations $\partial_x Q \pm \partial_t Q$ used for radiation boundary conditions are modified to:

$$0 = (e^{-at} + ax)\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial t} \tag{3}$$

$$0 = (e^{-at} - ax)\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial t} \tag{4}$$

We would like to apply outgoing radiation boundary conditions to our simulation. To determine how to correctly set up these conditions one can use Equation (2) to determine how the solutions theoretically behave. Taking the derivative with respect to time:

$$\frac{dx}{dt} = (1 - ax_0 - at)e^{-at}$$

2 a Positive

2.1 Theory

If $a > 0$, the te^{-at} term will make the "right moving" pulse initially move to the right. But e^{-at} will ultimately dominate over t, and so the pulse will change direction and move leftward. At the same time, the $x_0 e^{-at}$ term will make the pulse appear to contract. So the right moving pulse ultimately shrinks to zero width, and all its points go to $x = 0.0$ in infinite time. A point on the pulse that starts at $x = x_0$ changes its direction at time $t = 1/a - x_0$.

The "left moving" pulse behaves the same as the right moving pulse but in the opposite direction: it starts out moving leftward but changes direction, and contracts to a line as all points asymptote to x =0.0.

The symmetry between the behavior of both solutions means that in a way we apply the same boundary conditions on both sides of the domain of x. At first one uses Equation (4) on the right side and Equation (3) on the left side to prevent all inwardly moving pulses from entering through the boundaries. But we've shown that all outwardly moving pulses eventually become inwardly moving pulses when $a > 0$. So all parts of the right moving pulse that exits the right boundary eventually reenter. Likewise for the left moving pulse. The right edge x_0 =1.0 of the right moving pulse returns at a time t such that:

$$1.0 = (1.0 + t)e^{-at}$$

This value of t is the same instance when the left edge x_0 = -1.0 of the left moving pulse reenters our domain. So after that time t one needs boundary conditions that prevent **both** solutions from coming into the domain, from **either** boundary. One could possibly achieve this by switching to fixed boundaries after that time t.

2.2 Results

Movie # 1 shows a right moving pulse with β =1.0, a =0.3 and all the θ's =0.5. The pulse exhibits the predicted behavior, slowing down over time and contracting. The pulse also develops some distortion on its edges, and leaves a small left moving pulse when it exits the domain. Figure # 1 shows the convergence factor for this pulse. The Factor hovers around 4.0 before it drops at around time t =1.5, when the pulse is leaving. But it reverts to around 4.0 once the pulse has left. Movie # 2 shows a left moving pulse with the same parameters, and is a mirror image of the right moving pulse.

Unfortunately, the program fails to calculate a stable solution as one increases a. When one sets $a = 0.52$ using the above parameters, the condition number increase to the order of 100000, and the graph explodes at the boundaries at t =1.6. When one changes β to 2.0, as recommended by Alcubierre, and θ_t to 0.1, the condition number then decreases to a value of around 3.0 by the time t =2.0. But the graph **still** explodes at the boundaries, as shown in Figure # 2. I'm afraid that I have not found a good explanation for this explosion or a means of overcoming it. Also, the right edge of the pulse is moving leftward at t =2.0, but it only reaches x =1.06 by this time. So the fixed boundary conditions described above are not tested. For a >0.52, this problem at the boundaries happen at earlier times, and the condition numbers tend to increase to large values for the earliest time steps.

3 a Negative

3.1 Theory

Setting a <0.0 reverses some of the characteristics the pulses display with a positive a. Here, the $texp(-at)$ term causes the pulses to increase in speed exponentially, and the $x_0exp(-at)$ makes the pulses expand instead of contract. For the right moving pulse, the pulse's expansion leftward on its left side is initially greater than its movement rightward. So that part of the right moving pulse starts out moving left. Similarly, the right side of a left moving pulse starts out moving right. So one has to start out with boundary conditions that **permit** both solutions to work at both the boundaries. But at time $t = 1/a + 1.0$ the left edge x_0 =-1 of the right moving pulse (and the right edge of the left moving pulse) changes direction and so that part of the pulse starts moving back through the left boundary. By the time t such that $-1 = (-1+t)exp(-at)$ all of the right moving pulse has cleared the left boundary and is moving rightward. So for time t after this, one can simply use the standard radiation conditions of Equation (4) on the right boundary and Equation (3) on the left boundary.

3.2 Results

Movie #3 shows a right moving pulse with ρ =1.0, a =-0.5, and the θ's =0.5. This pulse, too, exhibits the characteristics predicted by theory: it accelerates and expands. Figure # 4 shows that the convergence factor starts at near 4.0 and gradually decreases, as it has done in previous examples, until it spikes between t =0.7 and 0.8 when the pulse has left the domain.

But one runs into problems as one decreases a. Setting a =-2.0 while keeping all the other parameters unchanged results in the condition number starting with order 10^{47}, and the solution exploding from the first step, as shown in Figure # 4. But when one sets β to 0.5 and θ_t to 0.1, one gets the stable right moving pulse shown in Movie # 4. The value of θ_t is particularly important, as its reduction to 0.1 lowers the condition number down to a range of 10-30. Different values of θ_f were also tried, but the value of θ_f proved virtually inconsequential to the results. Figure #5 shows the convergence factor for this pulse. The graph has a particularly large spike at around t =0.5 when the pulse exits, but the factor remains around 4.0 on both sides of the spike. And Movie # 5 shows a left moving pulse with the same parameters as in Movie #4, and it to is a mirror image of the right moving pulse.

For a =-2.0 one does have to worry about the left edge of the right moving pulse crossing the left boundary. I couldn't find an expression for boundary conditions that would let both solutions through on both sides, though. But for these particular pulses one could "cheat" because our right moving pulse is zero for most of its left side. So one could still use boundary conditions that only allowed left moving pulses at the left boundary, because those conditions are also satisfied by a right moving pulse whose parts that cross the left boundary are always zero.