Project #1: The 1-D Wave Equation

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1 Project01. f

For Project 1, I wrote the code Project01. f, which evolves the one-dimensional wave equation:

$$0 = \frac{d^2Q}{dx^2} - \frac{d^2Q}{dt^2}$$

and the code Converge. f which calculates the convergence factor for the solution.

$Q(x,t)$ is defined from $x = -1.0$ to $x = +1.0$. At time $t = 0.0$ Q is non-zero only for $x = [-0.2, 0.2]$. Project01. f has the option of using either of the following two functions for the non-zero domain.:

$$Q(x,0) = A * e^{(-\frac{x}{\Delta})^2} \quad (1)$$

$$Q(x,0) = 0.5 * (cos(\frac{\pi}{\Delta}x) + 1) \quad (2)$$

controlled by the variable Ngraph. One inputs the value of Δ in the file Project01. dat, along with the spacing dx and the timestep dt.

At present, Project01. f can take in grids of 3201 points along the x-axis ($dx = 0.000625$) and coarser versions. If one uses the finest grid, the program defines $Q(x,-dt)$ (Q_j^{-1}) as equal to $Q(x,0)$. For a coarser grid with $dx = Ndx0$, $dx0 = 0.000625$, Project01. f first evolves Qf on a 3201-point grid to step $N-1$ (Qf_j^{N-1}), where $(dt0/dx0) = (dt/dx)$. Then the program uses Qf_j^{N-1} to define Q_j^{-1}. This procedure is required so that the program will generate data that has the correct convergence factor.

2 Results

Once Q_j^{-1} is determined, the program calculates Q_j^n ($x = j * dx$, $t = n * dt$) using the three level recursion formula:

$$Q_j^{n+1} = 2 * Q_j^n - Q_j^{n-1} + (\frac{\Delta t}{\Delta x})^2 (Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n)$$

up to time $t = Time$ as input into Project01. dat. Figure 1 shows $Q(x,0)$ for Function (1) while Figure 7 shows $Q(x,0)$ for Function(2). In both cases, Q evolves such as to break into two pulses with half the original magnitude, going out towards the boundaries (Figures (2) and (7)). One chooses the type of boundary using the flag $Nbound$ in Project01. dat.

$$\begin{array}{rclcll}
Q(+1,t) & = & Q(-1,t) & & & Periodic \\
Q(-1,t) & = & Q(+1,t) & = & 0 & Fixed \\
0 & = & \partial_t Q(1,t) + \partial_x Q(1,t) & = & \partial_t Q(-1,t) + \partial_x Q(-1,t) & Radiation
\end{array}$$

Figures (3) and (9) show the results when the pulses hit the fixed boundaries. The pulses are reflected back, and their signs are reversed. But if the boundaries are periodic, the pulses reflect back with the same sign. Figures (4) and (10) show the pulses reflecting back on themselves. The radiation condition leads to the pulses propagating outside the boundaries leaving nothing between $x = [-1, 1]$, as shown in Figures (5) and (11).

The program has the option of printing out Q at a particular instant in time, input as $Timpt$ in Project01. dat, or for all timesteps up to time $= Time$, in order to make files for a gnuplot movie. The flag $Nmovie$ controls this option.

All the graphs use $(\Delta t/\Delta x) = 0.9$ But the graphs become unstable if one sets $(\Delta t/\Delta x) > 1.0$ Figures (6) and (12) show Q with $(\Delta t/\Delta x) = 1.1$, where at an early t the graph has huge values of Q that change considerably with each time step.

3 Convergence Factor

The program Converg. f takes as input the timestep dt, the spacing dx. and the value of $Time$. Converge. f then evolves Q up to $Time$ using a dx grid $Q4$, $dx/2$ grid $Q2$ and $dx/4$ grid $Q1$, using the procedure described above to calculate Q_j^{-1}. For each timestep n the program calulates:

$$norm42 = \sqrt{\frac{\sum_j (Q4_j^n - Q2_j^n)}{2 * Nx + 1}} \tag{3}$$

$$norm21 = \sqrt{\frac{\sum_j (Q2_j^n - Q1_j^n)}{4 * Nx + 1}} \tag{4}$$

where $Nx = 1/dx$, and prints out the convergence factor:

$$CF = norm42/norm21$$

into Factor. dat. The program has the option of which boundary and initial conditions to use.

Figures (13) - (15) uses data with $(\Delta t/\Delta x) = 0.5$. Figure (13) graphs the convergence factor for the periodic boundary. The factor stays close to four at all times except around $t = 1.0$ and $t = 2.0$. These times correspond to when the pulses have maximum constructive interference, and the factor soars to between 20 and 30. Figure (14) shows the factor for the fixed conditions. The graph also shows irregular behavior at $t = 1.0$ and $t = 2.0$, but not the large magnitude at $t = 1.0$, where the pulses experience maximum destructive interference. The Radiation condition (Figure (15)) has no extreme variation at any t. So all graphs show second-order convergence for all three boundary cases.