# Analytic Kerr Solution for Puncture Evolution

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Maximal slicing of a spacetime with a single Kerr black hole is analyzed. It is shown that for all spin parameters, a limiting hypersurface forms between the black hole horizon and the physical singularity. An analytic value for the limiting hypersurface, in terms of the Boyer-Lindqiuist coordinate $r$, is derived. Using these results, quasi-isotropic Kerr coordinates are used to show that a single singularity forms, located at the computational coordinate $\bar{r} = 0$, which corresponds to the limiting hypersurface. It is shown that in these coordinates, evolution of the initial data under maximal slicing remains stationary.

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Stationary solutions of the puncture method have been previously investigated for Schwarzschild black holes [1], [2].
Stationary solutions of the puncture method for Kerr black holes are derived here.

## I. PUNCTURE METHOD

The puncture method considered here is a version of BSSN evolution which uses *log + 1* slicing for evolution of the lapse

$$\partial_0 \alpha = -2\alpha K \tag{1}$$

and the *Gamma driver* equations for evolving the shift

$$\partial_0 \beta^i = \frac{3}{4}\bar{\gamma}^i_{jk}\bar{g}^{jk} \tag{2}$$

where $\bar{g}_{ij}$ denotes the conformal metric with the connection compatible with the conformal metric given by $\bar{\gamma}^i_{jk}$. The goal of equations (1) and (2) is to maintain the constraints

$$K = 0 \tag{3}$$
$$\bar{\Gamma}^i \equiv \bar{g}^{jk}\bar{\Gamma}^i_{jk} = 0 \tag{4}$$

The evolution equations for the lapse, equation (1), can be integrated in time to yield the constraint

$$\alpha\sqrt{g} = C \tag{5}$$

in cartesian coordinates with $C \in \mathbb{R}$ and spatial volume, $\sqrt{g}$, denoting the determinant of the full three metric. Maximal solutions are defined by the constraint $K = 0$, which, from equation (5), is equivalent to the constraint $\alpha\sqrt{g} = 1$. Only the slicing condition, equation (1), will have a direct effect on the derivations presented here.

## II. STATIONARY MAXIMAL PUNCTURE SOLUTIONS

Stationary maximal slices for Schwarzschild black holes in puncture coordinates initially explored by Hannam *et. al*, [2], using numerical simulations and later analytically by Baumgarte and Nauchlich, [1]. Expressing maximal slicing as $\alpha\sqrt{g} = 1$ facilitates the derivation and interpretation, showing that numerically stable stationary maximal slices in puncture evolutions form when a the physical singularity of the black hole is expressed in computational coordinates at single point, i.e. "puncture", which is numerically disconnected from the rest of the computational domain by the formation of a coordinate singularity. It is shown that this coordinate singularity forms inside the horizon of the black hole and outside of the physical singularity, allowing the puncture method to capture all physical properties of a given system which are causally connected to future timelike infinity, $i^+$.

### A. Schwarzschild

The Schwarschild metric in Schwarzschild coordinates is given by the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)\ dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}\ dr^2 + r^2\ d\theta^2 + r^2 sin^2\theta\ d\phi^2 \tag{6}$$

which can be expressed as

$$ds^2 = -\alpha_0^2 dt^2 + \frac{1}{\alpha_0^2} dr^2 + r^2 d\Omega^2 \tag{7}$$

with $\alpha_0 = \left(1 - \frac{2M}{r}\right)$ and $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$, which manifestly satisfies the constraint $\alpha\sqrt{g} = 1$. In Schwarzschild coordinates the radial coordinate, $r$, defines proper distance as measured by null geodesics since, from equation (6), $\frac{dr}{dt} = \pm 1$ for the ingoing and outgoing null rays. The Schwarzschild coordinates have a familiar coordinate singularity at $r = 2M$, at which the proper time of an object, as measured by an observer at $r > 2M$, remains fixed for all future measurements made by an observer at $r > 2M$. Through a coordinate change to isotropic coordinates, defined by

$$\begin{aligned} ds^2 &= -\left(\frac{1-\frac{M}{2\bar{r}}}{1+\frac{M}{2\bar{r}}}\right)^2\ dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4\ \left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right] \\ &= -\left(\frac{1-\frac{M}{2\bar{r}}}{\psi}\right)^2\ dt^2 + \psi^4\left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right] \end{aligned} \tag{8}$$

with

$$r = \bar{r}\left(1 + \frac{M}{2\bar{r}}\right)^2 = \psi^2 \bar{r} \tag{9}$$

the singularity at $r = 2M$ in Schwarzschild coordinates is removed, and so must be a coordinate singularity, as opposed to a physical singularity which can not be removed by a change in coordinates. This suggests that the coordinate singularity at $r = 2M$, which is located at the black hole horizon, can be shifted to a location inside of the horizion by rescaling the radial coordinate $r$. Since $r$ is the affine parameter of proper distance, introducing a shift term, $\beta^r$, will change $\frac{dr}{dt} = \pm 1$, allowing the coordinate singularity to be pushed inside of the horizon. Introducing a shift term of the form

$$\beta^R = \frac{C\alpha}{R^2} = \frac{C}{\sqrt{g}} \tag{10}$$

for constant $C \in \mathbb{R}$, the Schwarzschild line element, equation (6), becomes

$$ds^2 = -\left(\alpha^2 - \beta_R\beta^R\right) dt^2 - 2\alpha\beta_R\ dt\ dR + \frac{1}{\alpha^2} dR^2 + R^2 d\Omega^2 \tag{11}$$

with the new radial coordinate, $R\,(r, \beta_R)$, lapse, $\alpha\,(r)$, and shift, $\beta^R\,(r)$, being functions of the Schwarzschild coordinate $r$. Since the horizon is defined in Schwarzschild coordinates by the metric component $g_{00} = -\left(1 - \frac{2M}{r}\right) = 0$, the new lapse must satisfy

$$\alpha^2 + \beta_R\beta^R = \alpha^2 + \frac{C^2}{R^4} = \alpha_0^2 = \left(1 - \frac{2M}{r}\right) \tag{12}$$

at $r = 2M$. In order for the coordinate change to have a smooth dependence on the choice of constant $C$, and satisfy the asymptotic boundary condition $R\,(r) \to r$ as $r \to \inf$, the simplest form that the lapse $\alpha$ can take will be

$$\alpha^2 = 1 - \frac{2M}{R} + \frac{C^2}{R^4} \tag{13}$$

so that $R = r$ when $C = 0$. Because of the constraint imposed to ensure maximal slicing, $\alpha\sqrt{g} = 1$, changes in the temporal coordinate, $t$, must be uniquely specified by changes in the determinant of the spatial metric, which for equation (11) is completely determined by the radial coordinate, $R$. Equating the spatial components of equation (11) to the spatial components of the isotropic metric, equation (8), yields

$$\alpha^{-2} dR^2 + R^2 d\Omega^2 = \psi^4\left[d\bar{r}^2 + \bar{r}^2 d\Omega^2\right] \tag{14}$$

Since the coordinates $\theta$ and $\phi$ remain unchanged, equation (14) can only be satisfied when

$$\frac{dR}{\alpha} = \psi^2 d\bar{r} \tag{15}$$

$$R = \psi^2 \bar{r} \tag{16}$$

with $\psi = 1 + \frac{M}{2\bar{r}}$, a function of $\bar{r}$, and $\alpha$ a function of $R$. Inserting in the results of equation (13), an expression for $R$ as a function of $\bar{r}$, $R(\bar{r})$, can be found by integrating

$$\int \frac{1}{R} \frac{dR}{\sqrt{1 - \frac{2M}{R} + \frac{C^2}{R^4}}} = \int \frac{1}{\psi^2 \bar{r}} \psi^2 d\bar{r} \tag{17}$$

which simplifies to

$$\int \frac{RdR}{\sqrt{R^4 - 2MR^3 + C^2}} = \int \frac{d\bar{r}}{\bar{r}} \tag{18}$$

The integral $\int \frac{d\bar{r}}{\bar{r}}$ has a singularity at $\bar{r} = 0$, at which the integrand, $\frac{1}{\bar{r}}$, is not analytic. Since this is the only singularity, the solution space will be analytic in the branch $\bar{r} > 0$. Extending the real valued function $\bar{r}$ to the complex domain by setting $z = \bar{r}$, the integral along any closed contour enclosing $z = 0$ will be

$$\oint_C \frac{dz}{z} = 2\pi i \tag{19}$$

whence the singularity at $z = 0$ is a simple pole. Since $\frac{1}{\bar{r}}$ is analytic in the domain $\bar{r} > 0$, the integrand on the left hand side of equation (18), $\frac{R}{\sqrt{R^4 - 2MR^3 + C^2}}$, must also be analytic everywhere some domain $R > b$ for a constant $b \in \mathbb{R}$, and must have a singularity at $R = b$ yielding a simple pole. Since the map from the domain of $\bar{r}$ to the domain of $R$ is found by integrating equation (18), the integrands must satisfy the same analycity requirements in order for the Jacobian of the coordinate transformation, $\frac{dR}{d\bar{r}}$, to be an isomorphism. The requirement that the pole at $R = b$ have the same order as the pole $\bar{r} = 0$ is necessary in order for the polynomial expansion of real functions in the domains of $R$ and $\bar{r}$ to converge in the vicinity of the singularities at $R = b$ and $\bar{r} = 0$ respectively, a necessary requirement for the Taylor expansion of real functions to yield polynomial expressions which converge. These requirements will be met whenever the constant $C$ yields a real double root of the quartic function

$$(R - b)\sqrt{R^2 + n_1 R + n_0} = \sqrt{R^4 - 2MR^3 + C^2} \tag{20}$$

with constants $n_1, n_0 \in \mathbb{R}$ which depend on $C$. In order for $R$ to be analytic in the domain $R > b$, the constant $C$ must also be chosen such that

$$|b| > \left| \frac{-n_1 \pm \sqrt{(n_1)^2 - 4n_0}}{2} \right| \tag{21}$$

in order for all singularities in the complex domain to be the closed disk, $U$, defined by $U = \{z \mid z\bar{z} \leq b^2\}$, otherwise $\frac{R}{(R-b)\sqrt{R^2+n_1R+n_0}}$ may not be analytic in the complex domain defined by $R > b$. For the Schwarzschild black hole

$$b = \tfrac{3M}{2} \tag{22}$$

$$C = \tfrac{3\sqrt{3}M^2}{4} \tag{23}$$

in agreement with the results found in [1] and [2]. A general method for finding the necessary constant $C$, and the resulting singularity at $b$, is given in appendix (A).

### B. Kerr

The line element of the standard Kerr metric in Boyer-Lindquist coordinates is given by

$$ds^2 = -(1 - \frac{2Mr}{\rho^2})\ dt^2 - \frac{4Marsin^2\theta}{\rho^2}\ dtd\phi \tag{24}$$
$$+\frac{\Sigma}{\rho^2} sin^2\theta\ d\phi^2 + \frac{\rho^2}{\Delta}\ dr^2 + \rho^2\ d\theta^2$$

where

$$\rho^2 = r^2 + a^2 cos^2\theta \tag{25}$$
$$\Delta = r^2 - 2Mr - a^2 \tag{26}$$
$$\Sigma = (r^2 + a^2)^2 - a^2 \Delta sin^2\theta \tag{27}$$

The constant $M \in \mathbb{R}$ defines the mass, and the constant $a \in \mathbb{R}$, for $0 \leq a < M$, defines the angular momentum per unit mass. Setting $a = 0$ in equation (24) yields the Schwarzschild metric, equation (6), whence the Boyer-Lindquist coordinates must be an extension of Schwarzschild coordinates to include black holes with constant angular momentum, $J = aM$. In Boyer-Lindquist coordinates, the stationary maximal solutions for Kerr black holes to be found, and interpreted, in a method similar to the Schwarzschild case, subsection (II A).

From equation (24), the lapse is defined as

$$\alpha_0^2 = \frac{\rho^2 \Delta}{\Sigma} \tag{28}$$

with the determinant of the spatial three metric being

$$g = \frac{1}{\alpha_0^2} \rho^4 sin^2\theta \tag{29}$$

In Boyer-Lindquist coordinates, the determinant of the coordinate transformation from cartesian coordinates is $\rho^4 sin^2\theta$, therefore the spacial slicing for the Kerr metric defined in equation (24) will be maximal in cartesian coordinates, satisfying $\alpha_0 \sqrt{g} = 1$. Because the Kerr solution is independent of the coordinates $t$ and $\phi$, the spacetime posesses a temporal as well as an azimuthal symmetry. As a result, in any coordinate system $X^A$ there must exist Killing vectors, corresponding to translation along $t$ and $\phi$, with components

$$\xi^A_{(t)} \equiv \frac{dX^A}{dt} \tag{30}$$
$$\xi^A_{(\phi)} \equiv \frac{dX^A}{d\phi} \tag{31}$$

with a general Killing vector satisfiying

$$\mathcal{L}_{\vec{\xi}} g_{AB} = \nabla_A \xi_B + \nabla_B \xi_A = 0 \tag{32}$$

for Kerr metric $g_{AB}$ and corresponding metric compatible connection, $\nabla_A$. Any coordinate change must preserve these symmetries.

As in equation (7) for the Schwarzschild metric, the Kerr metric, equation (24), can be expressed as

$$ds^2 = -\left(\alpha_0^2 - \beta^\phi \beta_\phi\right)\, dt^2 - 2\beta_\phi\, dt\, d\phi + \frac{1}{\alpha_0^2}\left(\frac{\rho^4}{\Sigma}\right) dr^2 + \rho^2\, d\theta^2 + \left(\frac{\Sigma}{\rho^4}\right) \rho^2 sin^2\theta\, d\phi^2 \tag{33}$$

with the shift vector, $\beta^\phi$, defined by

$$g_{t\phi} = \beta^\phi g_{\phi\phi} = \beta_\phi \equiv -\frac{2Mar sin^2\theta}{\rho^2} \tag{34}$$

Using equations (24), (30), and (32), in Boyer-Lindquist coordinates, the shift vector, $\vec{\beta} = \left(0, 0, \beta^\phi\right)$, satisfies

$$g_{t\phi} \frac{dt}{dt} = g_{\phi\phi} \frac{d\phi}{dt} \tag{35}$$

and so define how the coordinate position $\phi$ must change with a change in coordinate time. Allowing reparameterizations of coordinate time, the most general Killing vector, $\vec{\xi}$, will be

$$\begin{aligned} \vec{\xi} &= \left(\tfrac{\alpha_0}{\alpha},\ 0,\ 0,\ -\left(\tfrac{\alpha_0}{\alpha}\right) \tfrac{\beta^\phi}{\alpha}\right) \\ &= \left(\tfrac{\alpha_0}{\alpha},\ 0,\ 0,\ \left(\tfrac{\alpha_0}{\alpha}\right) \tfrac{2Mar}{\Sigma}\right) \end{aligned} \tag{36}$$

which dictates how $\beta^\phi$ must transform under a coordinate transformation rescaling the Boyer-Lindquist time coordinate. Limiting permissible coordinate transformations to those which satisfy the constraint $\alpha\sqrt{g} = 1$, stationary maximal slices for Kerr black holes will be uniquely determined by the components of the spatial metric

$$ds^2_{(3)} = \frac{1}{\alpha^2}\left(\frac{\rho^4}{\Sigma}\right) dR^2 + \rho^2\ d\theta^2 + \left(\frac{\Sigma}{\rho^4}\right)\rho^2 sin^2\theta\ d\phi^2 \tag{37}$$

Because of the dependence of the solution on both Boyer-Lindquist coordinates $r$ and $\theta$, it is important to note that, in general, the coordinate $R$ in equation (37) will be a complex function of both $r$ and $\theta$.

The Kerr metric in Boyer-Lindquist coordinates, equation (24), will have singularities at $\Delta = 0$ and $\rho^2 = 0$. Comparing these singularities with the singularities of the Schwarzschild metric in the limit $a \to 0$, the singularities at $\Delta = 0$ must be coordinate singularities, while the singularities at $\rho^2 = 0$ must be physical. The coordinate singularity $\Delta = 0$ has two solutions

$$r_\pm \equiv M \pm \sqrt{M^2 - a^2} \tag{38}$$

which define the event horizon, located at $r_+$, and the inner apparent horizon, located at $r_-$. The physical singularity, $\rho^2 = 0$, in the domain $0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi, r \geq 0$, can only be located at $r = 0$ in the equatorial plane, defined by $\theta = \frac{\pi}{2}$, and must be a ring singularity due to azimuthal symmetry.

Extending the isotropic coordinates for Schwarzschild black holes, equation (8), to Kerr black holes yields quasi-isotropic coordinates [3], defined by the line element

$$ds^2 = -\left(\frac{\rho^2\Delta}{\Sigma} - \beta^\phi\beta_\phi\right) dt^2 - 2\beta_\phi\ dt d\phi + ds^2_{(3)} \tag{39}$$

$$ds^2_{(3)} \equiv \chi^4\left[\left(\frac{\rho^4}{\Sigma}\right)^{1/3}\left(d\bar{r}^2 + \bar{r}^2 d\theta^2\right) + \bar{r}^2 sin\theta^2\left(\frac{\Sigma}{\rho^4}\right)^{2/3} d\phi^2\right] \tag{40}$$

where

$$\rho^2 = \psi^4\bar{r}^2 + a^2cos^2\theta \tag{41}$$

$$\Delta = \psi^4\bar{r}^2 - 2M\psi^2\bar{r} + a^2 \tag{42}$$

$$\Sigma = \left(\psi^4\bar{r}^2 + a^2\right)^2 - a^2\Delta sin^2\theta \tag{43}$$

$$\beta^\phi = -\frac{2M\psi^2\bar{r}a}{\Sigma} \tag{44}$$

$$\chi^4 = (\frac{\rho^2\Sigma}{\bar{r}^6})^{1/3} \tag{45}$$

where the coordinate transformation from quasi-isotropic coordinate $\bar{r}$ to Boyer-Lindquist coordinate $r$ is defined by

$$r \equiv \psi^2\bar{r} = \bar{r}\left(1 + \frac{M+a}{2\bar{r}}\right)\left(1 + \frac{M-a}{2\bar{r}}\right) \tag{46}$$

The angular coordinates $\theta, \phi$ remain unchanged.

In quasi-isotropic coordinates, the coordinate singularities $r_\pm$, equation (38), are located at

$$\bar{r}_\pm \equiv \pm\frac{\left|\sqrt{M^2 - a^2}\right|}{2} \tag{47}$$

Only the event horizon is located accessible in the region $\bar{r} > 0$, but will no longer be a coordinate singularity since the volume element on the spatial manifold, $\sqrt{g} = \chi^6$, is nowhere vanishing in the domain $0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi, \bar{r} > 0$. Using equation (46), the physical singularity $\rho^2 = 0$,, located at $r = 0, \theta = \frac{\pi}{2}$ in Boyer-Lindquist coordinates, will correspond to solutions of

$$\bar{r}^2 + \bar{r}M + \frac{M^2 - a^2}{4} = 0 \tag{48}$$

in the plane $\theta = \frac{\pi}{2}$, which are

$$\bar{r} = \frac{-M \pm a}{2} \tag{49}$$

Whenever the spin parameter, $a$, takes values in the range $0 \leq a < M$, no solutions to equation (49) can be located in the domain $0 \leq \phi < 2\pi, 0 \leq \theta \leq \pi, \bar{r} > 0$. Using equations (46) and (47), the only singularity in the quasi-isotropic coordinate system will be located inside of the event horizon at $\bar{r} = 0$,

As noted above, a general coordinate change affecting the coordinate time, $t$, and radial coordinate, $r$, will have a complicated dependence on $\theta$. In order to avoid having to find the complex analytic expression for stationary maximal slices of Kerr space times in general, it will be easier to consider surfaces of constant $\theta$ for which a shift vector of the form

$$\beta^R = \frac{C}{\sqrt{g}} \tag{50}$$

with constant $C \in \mathbb{R}$, defines an isomorphism from the surface of constant $\theta$ in Boyer-Lindquist coordinates, $(r, \theta, \phi)$, to a surface of constant $\theta$ in the new coordinate system, $(R, \theta, \phi)$. This will be possible whenever the surface of constant $\theta$ is chosen so that all geodesics tangent to the surface at any $r$ remain tangent to the plane for all $r$, in which case a shift component of the form given in equation (50) will have the effect of rescaling the coordinate time $t$ while leaving $g_{\theta\theta}\ d\theta^2$ fixed. Examining the null geodesics of the Kerr metric in Boyer-Lindquist coordinates, the surfaces of interest will be the equatorial plane, $\theta = \frac{\pi}{2}$, and along the axis of rotation, $\theta = 0$ and $\theta = \pi$.

As in equation (37), the Kerr metric in Boyer-Lindquist coordinates can be written in the form

$$ds^2_{(3)} = \frac{1}{\alpha^2}\frac{\rho^4}{\Sigma}\ dr^2 + \rho^2\ d\theta^2 + \frac{\Sigma}{\rho^2} sin^2\theta\ d\phi^2 \tag{51}$$

with $\alpha = \alpha_0$. After restricting to either the equatorial plane, by setting $\theta = \frac{\pi}{2}$, or axis of rotation, by setting $\theta = 0$ or $\theta = \pi$, the introduction of a radial shift component of the form given in equation (50) will modify the lapse, $\alpha$, by

$$\beta_r \beta^r = \frac{C^2}{\Sigma} \tag{52}$$

Using equation (28), the modified lapse will be given by

$$\alpha^2 = \frac{\rho^2\Delta}{\Sigma} + \frac{C^2}{\Sigma} \tag{53}$$

In the new coordinate system, the spatial metric, equation (51), becomes

$$ds^2_{(3)} = \frac{\rho^4}{\rho^2\Delta + C^2}\ dR^2 + \rho^2\ d\theta^2 + \frac{\Sigma}{\rho^2} sin^2\theta\ d\phi^2 \tag{54}$$

with $r$ replaced by $R$ in $\rho, \Delta, \Sigma$. As was done for the Schwarzschild solution in subsection (II A), requiring the spatial metric of equation (54) to be equivalent to the spatial metric in quasi-isotropic coordinates, equation (39), yields

$$\frac{\rho^4}{\rho^2\Delta + C^2} dr^2 + \rho^2\ d\theta^2 + \frac{\Sigma}{\rho^2} sin^2\theta\ d\phi^2 = \chi^4 \left[ \left(\frac{\rho^4}{\Sigma}\right)^{1/3} \left(d\bar{r}^2 + \bar{r}^2 d\theta^2\right) + \bar{r}^2 sin\theta^2 \left(\frac{\Sigma}{\rho^4}\right)^{2/3} d\phi^2 \right] \tag{55}$$

Since $\theta$ remains constant, the coefficients of $d\theta^2$ must be equivalent, yielding the relation

$$\rho^2 = \chi^4 \left(\frac{\rho^4}{\Sigma}\right)^{1/3} \bar{r}^2 \tag{56}$$

Using equation (41), the coefficients of $d\phi^2$ are trivially equivalent. Inserting these results into equation (55), yields

$$\int \frac{\rho}{\sqrt{\rho^2\Delta + C^2}}\ dR = \int \frac{d\bar{r}}{\bar{r}} \tag{57}$$

As in subsection (II A), in order for solutions to equation (57) to define an isomorphism from $\bar{r} > 0$ to $R > b$ requires that the constant $C$ yield a real double root for the quartic equation $\rho^2\Delta + C^2$ which is located at $R = b$. Using the method described in appendix (A) with coefficients

$$a_3 = -2M \tag{58}$$

$$a_2 = a^2 \left(cos^2\theta + 1\right) \tag{59}$$

$$a_1 = -2Ma^2 cos^2\theta \tag{60}$$

$$a_0 = a^4 cos^2\theta + C^2 \tag{61}$$

yields double roots in the equatorial plane at

$$b = \left[ \frac{3M \mp \sqrt{9M^2 - 8a^2}}{4} \right] \tag{62}$$

The double roots along the axis of rotation are more complex and far less informative, since the physical singularity resides exclusively in the $\theta = \frac{\pi}{2}$ plane, and so are detailed in appendix (B).

The main result is that in quasi-isotropic coordinates, a stationary maximal slice exists with a single singularity located at $\bar{r} = 0$ which corresponds to a coordinate singularity in Boyer-Lindquist coordinates located at

$$r_b = \frac{3M + \sqrt{9M^2 - 8a^2}}{4} \tag{63}$$

satisfying

$$r_- < r_b < r_+ \tag{64}$$

for all $0 \leq a < M$. This result agrees with other calculations of the radial location of throat formation for maximally sliced stationary kerr spacetimes, [4], derived using alternative methods.

### C. Conclusion

In much the same way that the coordinate singularity in the original Schwarzschild coordinates means that an outside observer measures infinite coordinate time for a test particle to make it to the horizon, the log+1 slicing scales the affine parameter so that a coordinate singularity with a real double root forms inside of the horizon. The double root means that the relation between the well known Kerr radial coordinate and the radial component of the evolved puncture system has a branch point at the coordinate singularity. Demanding that the coordinate change from $R$ to $\bar{r}$ be analytic requires the function under the root in denominator of the left hand side of equation (18) for the Schwarzschild case, or equation (57) for the Kerr case, to have a double root, thereby fixing the radial component of the shift, $\beta^R$.

## APPENDIX A: REAL DOUBLE ROOT OF A QUARTIC

In order for a function $f(r) = (r-b)^2$ to divide the quartic function $g(r) = r^4 + a_3 r^3 + a_2 r^2 + a_1 r^1 + a_0 r^0$ and have zero remainder, $b$ must solve

$$\begin{aligned} r^2 - 2br + b^2 \mid r^4 + a_3 r^3 + a_2 r^2 + a_1 r^1 + a_o \\ = (r^2 - 2br + b^2)(r^2 + n_1 r + n_o) + 0 \end{aligned} \tag{A1}$$

where

$$n_1 = (a_3 + 2b) \tag{A2}$$

$$n_0 = a_2 - b^2 + (a_3 + 2b)\, 2b \tag{A3}$$

Expressed in terms of the coefficients $a_3, a_2, a_1, a_0$, the double root $b$ must simultaneously solve the equations

$$\left[a_2 - b^2 + (a_3 + 2b)\, 2b\right] 2b = b^2 (a_3 + 2b) - a_1 \tag{A4}$$

which is independent of $C$, and

$$\left[a_2 - b^2 + (a_3 + 2b)\, 2b\right] b^2 = a_o \tag{A5}$$

which will be dependent upon $C$. Use equation (A4) to solve for possible values for $b$, and equation (A5) to determine the corresponding value for $C$.

## APPENDIX B: DOUBLE ROOT ON AXIS OF ROTATION

Maple returns the following value for $b$ on along the axis of rotation,

$$b = \frac{\frac{3}{2}M^2 - 2a^2}{\sqrt[3]{54\,M\,a^2 + 27\,M^3 + 6\,\sqrt{48\,a^6 - 27\,a^4 M^2 + 162\,a^2 M^4}}} + \frac{M}{2} + \frac{\sqrt[3]{54\,M\,a^2 + 27\,M^3 + 6\,\sqrt{48\,a^6 - 27\,a^4 M^2 + 162\,a^2 M^4}}}{6} \tag{B1}$$


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