The Lagrangian method of finding apparent horizons via the mean curvature method implements a moving coordinate system. The system is moved according to the equations of motion derived in terms of a parameterized moving front. Numerical schemes are based upon discrete approximations of the equations of motion. In this moving coordinate system, the particles describing the position of the coordinates will move according to

$$\dot{\vec{x}} = \kappa \hat{n}, \tag{0.1}$$

where $\hat{n}$ is the normal to the surface of interest (apparent horizon), κ is the curvature, and $\dot{\vec{x}}$ is the velocity of the front. By definition, the $\dot{\vec{x}}$, upon reaching the apparent horizon, equals zero since at that point $\kappa = 0$. The surface will stop propagating.

First the equations of motion are defined for a smooth, initially closed curve in two dimensions (for simplicity of statement) called $\gamma(0)$. A series of curves defined as $\gamma(t)$ and generated by moving $\gamma(0)$ along its normal vector with speed F(κ). A natural parameterization of the curve γ at time t is a position vector, $\vec{x}$(s,t). The s, $0 \leq s \leq S$, is restricted to boundary conditions

$$\vec{x}(0,t) = \vec{x}(S,t). \tag{0.2}$$

The normal and the curvature are related in terms of the parameter s as

$$\hat{n} \cdot \vec{x}_t = F(\kappa), \text{ where } \hat{n}(s,t) \text{ and } \vec{\kappa}(s,t) \tag{0.3}$$

are the outward normal and curvature respectively. With $\vec{x}$ = (x,y) we have the components:

$$x_t = F\left[\frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2 + y_s^2)^{3/2}}\right]\left(\frac{y_s}{(x_s^2 + y_s^2)^{1/2}}\right), \tag{0.4}$$

$$y_t = F\left[\frac{y_{ss}x_s - x_{ss}y_s}{(x_s^2 + y_s^2)^{3/2}}\right]\left(\frac{-x_s}{(x_s^2 + y_s^2)^{1/2}}\right). \tag{0.5}$$

A closed curve such as γ will shrink under its own curvature, collapsing to a point if F(κ) = -κ. Any oscillations in the initial curve decay depending on the sign of dF/dt at κ=0. This test of the curvature method came from a paper discussing propagating interfaces such as flame [Sethian, 1990]. A

curve can be introduced to a space-time region containing a black hole system and allowed to relax into the apparent horizon. The relaxation obeys laws similar to those governing flame propagation and crystal formation, namely the Hamilton-Jacobi equation and hyperbolic conservation law. The horizon can be seen as an interface that depends on local curvature (which depends on divergence). Particles defining an initial curve are given velocities equal to the local curvature. As the particles propagate in a curve with the local curvature, they will stick to the apparent horizon where there velocity will go to zero within some limit (Sethian, 1990).

The location of the apparent hroizons has been tested for the one and two black holes cases using this method with Eddington-Finkelstein coordinates in the Schwarzschild metric. The results of these tests are contained in the previous page.