FOUR

In Search of a Solution

In the previous chapter the field equations for the axially symmetric, rotating perfect fluid were reduced with a similarity transformation to a complicated set of coupled, nonlinear, second order, ordinary differential equations. Due to the complex nature of these equations, a complete solution of these equations could not be found. This chapter describes some attempts made to simplify and solve the equations. First, some conditions on the metric components are discussed, and the field equations are evaluated for consistency. Secondly, the equations are expanded in a power series about $\eta = 0$, which determines the parameters α and β. The equations are solved for an highly specialized case where one assumes that the unknown functions are all powers of η. This section concludes by noting some ideas that did not lead to immediate results, and possible avenues for further research.

4.1 Metric Constraints

Ideally, a solution of the field equations could be found that meets all the boundary conditions required for a physical interpretation. While the boundary conditions cannot be fully applied until the solution is known, they do supply some general constraints that can indirectly aid in the solving process. For example, because the metric describes the spacetime inside a stellar object, one expects the metric components to be well-behaved throughout the matter distribution. In this case, well-behaved functions are continuous, smooth, differentiable, and free of singularities. This condition restricts the general properties of the unknown functions, but adds no additional mathematical expressions that might directly aid in the solving process. This section examines conditions that can supply more information about the metric functions, and closes with a check for mathematical consistency.

One physical condition results from the equation of state and special relativity. The perfect fluid's equation of state, $p = \theta\mu$, enters the field equations through the parameter λ, introduced in equation (3.32)

$$\lambda = \frac{1}{4}\left(\frac{1}{\theta} + 3\right). \tag{4.1}$$

As discussed earlier in Chapter Three, special relativity restricts θ to the interval $0 < \theta \leq 1$ by requiring that the speed of sound in the medium is less than or equal to the speed of light. This restriction on θ corresponds to restricting λ to the interval $\lambda \geq 1$. In the vacuum case there is no such restriction, and λ is a free parameter.

The similarity transformation that reduced the field equations to ordinary differential equations introduced four new functions of η: A, B, C and E. The metric (2.20)

$$ds^2 = -f(dt + Q\,d\phi)^2 + \frac{1}{f}\left[s^2 d\phi^2 + e^{2\zeta}(d\rho^2 + dz^2)\right] \tag{4.2}$$

can be expressed in terms of the new functions, and becomes

$$\begin{aligned} ds^2 = -\rho^\beta C\left[dt - \frac{\rho^{\alpha-\beta}}{(\alpha-\beta)}\frac{B}{\eta C^2}\left[\beta E + \eta(\eta^2+1)E'\right]d\phi\right]^2 \\ + \frac{1}{\rho^\beta C}\left[\rho^{2\alpha}B^2 d\phi^2 + \rho^{2(\lambda\beta-1)}e^{2A}(d\rho^2 + dz^2)\right]. \end{aligned} \tag{4.3}$$

Furthermore, the perfect fluid's mass density and pressure can also be written in terms of the new functions. From equations (3.32–34), the mass density is

$$\mu = \kappa(\rho^\beta C)^{1-2\lambda} \tag{4.4}$$

and the pressure is

$$p = \kappa\theta(\rho^\beta C)^{1-2\lambda}. \tag{4.5}$$

The metric and the expressions for μ and p each contain the product $\rho^\beta C$. This product and its inverse must be finite as $\rho \to 0$ to avoid any singularities in the metric or infinite densities or pressures in the matter distribution. This requires that

$$\lim_{\rho\to 0} C \propto \rho^{-\beta}. \tag{4.6}$$

If the limit is taken at a finite value for z, then the limit can be expressed in terms of η

$$\lim_{\eta\to 0} C \propto \eta^{-\beta}. \tag{4.7}$$

The relation in equation (4.6) will be assumed to be valid to avoid singularities on the rotation axis. This assumption will not always hold—in fact one simple solution at the end of this chapter requires that C = constant—but the assumption will be maintained as long as possible.

One more physical restriction will be mentioned here, the condition of elementary flatness. This condition requires that the ratio of a circle's circumference to its radius is 2π, as the circle's radius becomes vanishingly small.[55] Singularities of various types often occur along the rotation axis, and thus the axis is a natural place to apply this condition. For simplicity, the notation used here is that of the metric in equation (4.2). The circumference of a circle centered on the rotation axis is obtained by integrating the proper length along the arc that defines the circle. Along this arc, $d\rho$, dz and dt all equal zero. The proper length integral becomes

$$\text{circumference} = \int_0^{2\pi} \sqrt{g_{\phi\phi}}\, d\phi \tag{4.8}$$

$$= \int_0^{2\pi} \sqrt{\frac{s^2}{f} - fQ^2}\, d\phi. \tag{4.9}$$

The metric functions s, f and Q are functions of ρ and z only. The allows the simple integration

$$\text{circumference} = 2\pi\sqrt{\frac{s^2}{f} - fQ^2}. \tag{4.10}$$

[55] Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland Publishing Company) pp. 313–314.

The radius is obtained in a similar manner

$$\text{radius} = \int_0^\rho \sqrt{g_{\rho\rho}}\, d\rho \tag{4.11}$$

$$= \int_0^\rho \frac{e^\zeta}{\sqrt{f}}\, d\rho. \tag{4.12}$$

Here, however, the metric components s, f and Q are functions of ρ, the integration variable. If the circle's radius becomes infinitesimally small, so that the integration is over a very small area in space, then these metric functions are approximately constant in the integral. This limit also allows a simple integration for the radius:

$$\lim_{\rho\to 0} \text{radius} = \lim_{\rho\to 0} \frac{e^\zeta}{\sqrt{f}}\rho. \tag{4.13}$$

Equations (4.10) and (4.13) can then be combined to express the elementary flatness condition as[56]

$$\lim_{\rho\to 0} \frac{\sqrt{s^2 - f^2Q^2}}{\rho e^\zeta} = 1. \tag{4.14}$$

Equation (4.14) will hold if the spacetime around the rotation axis is locally flat. This condition will not hold if the region contains any singularities.

A mathematical condition that the equations must satisfy is that the equations are internally consistent. This can be checked by combining all six equations into a single expression. If the field equations are over-determined, this single expression will contain any conditions that the metric coefficients A, B, C, and E must satisfy to give a self-consistent set of equations. The fourth field equation, equation (3.60), was already used to eliminate the $C^{-\sigma}$ terms to produce the equations (3.63–68). Therefore, the fourth field equation will not be included in this analysis. Equation (3.68) can be subtracted from (3.67) to eliminate the A'' term. Then the B'' can be eliminated with

[56] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press). The general form of this condition is given on page 192. The condition is specialized for perfect fluids on page 217 for the metric on page 215.

equation (3.65). This combination of four field equations yields a single, first order differential equation

$$\begin{aligned} &-\alpha(\alpha+1) - \frac{1}{2}\beta^2 + 2\lambda\alpha\beta - \frac{1}{2}\beta^2\frac{E^2}{C^2} \\ &\quad + \frac{1}{2}\eta(\eta^2+1)\left[2(\lambda\beta-\alpha)\frac{B'}{B} + 2\alpha A' - \beta\frac{EE'}{C^2} - \beta\frac{C'}{C}\right] = 0. \end{aligned} \tag{4.15}$$

Equation (4.15) can now be used to create a new first order equation. This is accomplished by taking the derivative of (4.15) with respect to η, and then using the field equations to eliminate all second derivative terms. For example, equation (3.63) expresses E'' in terms of first derivatives only. Equation (3.64) is a similar expression for C''. Equation (3.65) contains an expression for B'' and (3.67) expresses A'' with first derivatives only. This new first order equation is

$$\begin{aligned} &-\alpha + \left[\lambda\beta\eta(\eta^2+1) - \alpha\eta^3 + (\alpha\lambda\beta)\eta^2(\eta^2+1)\frac{B'}{B}\right]\frac{B'}{B} \\ &\quad + \left[1 - \eta\frac{B'}{B}\right]\left[2\lambda\alpha\beta - \alpha^2 - \frac{1}{2}\beta^2\left(1 + \frac{E^2}{C^2}\right)\right. \\ &\quad \left. + \eta^2(\eta^2+1)\left[\alpha A' - \frac{1}{2}\beta\left(\frac{C'}{C} + \frac{EE'}{C^2}\right)\right]\right] = 0. \end{aligned} \tag{4.16}$$

Equation (4.15) can be solved for A', and when the resulting expression is substituted into (4.16), the expression (4.16) becomes simply

$$0 = 0. \tag{4.17}$$

Thus when the field equations are combined into a single expression, no condition equation remains behind, and the equations are self-consistent. As mentioned before, at the end of § 3.5, this indicates that the field equations are redundant and not over-determined.

4.2 A Power Series

The parameters α and β used in the finite Lie-Bäcklund transformation appear in the reduced field equations. These parameters are completely arbitrary in the symmetry transformations; however, when substituted into the field equations these parameters may need to adopt specific values. One way to extract information from the field equations without solving them is to examine their behavior as $\eta \to 0$ by expanding the equations in a power series about $\eta = 0$. A general polynomial of an arbitrary parameter x

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 \tag{4.18}$$

can only equal zero if every coefficient a_i $(i = 1, \ldots, n)$ equals zero: $a_i = 0$. By writing the field equations as a power series, this property can be exploited to determine α and β.

The functions B, C, and E appear in the field equations only in ratios such as B'/B or EE'/C^2. Because these combinations appear so frequently, the arbitrary ratio X/Y will be expanded here, where X and Y can stand for any function or derivative (B, B', B'', C, $\ldots$) The expansion is performed about $\eta = 0$

$$\begin{aligned} \frac{X}{Y} = \frac{X}{Y}\bigg|_{\eta=0} &+ \left(\frac{X'}{Y} - \frac{XY'}{Y^2}\right)\bigg|_{\eta=0} \eta \\ &+ \frac{1}{2}\left(\frac{X''}{Y} - 2\frac{X'Y'}{Y^2} + 2\frac{XY'^2}{Y^3} - \frac{XY''}{Y^2}\right)\bigg|_{\eta=0} \eta^2 + O(\eta^3) \end{aligned} \tag{4.19}$$

Complicated terms such as EE'/C^2 can be handled by expanding first E/C and then multiplying by the expansion for E'/C, and then collecting powers of η. The function A and its derivatives do not appear in ratios. The expansions of these functions are very simple. For example, A' is expanded in the following manner

$$A' = A'\Big|_{\eta=0} + A''\Big|_{\eta=0}\eta + \frac{1}{2}A'''\Big|_{\eta=0}\eta^2 + O(\eta^3). \tag{4.20}$$

Maple was used to perform the expansion of the field equations and collect the coefficients of η up to second order. The coefficients of each term were then set to zero to produce 18 equations. Four equations that result form the η^0 coefficients are of particular interest. (The other fourteen equations do not give additional pertinent information.) In the following equations, the functions C and E evaluated at $\eta = 0$ are denoted as $C(0)$ and $E(0)$ respectively. First, from the first field equation

$$\beta(\alpha - \beta - 1)E(0) = 0. \tag{4.21}$$

The second field equation gives that

$$(\alpha - 1)(\beta - 2\lambda\alpha) + \beta^2 \frac{E(0)^2}{C(0)^2} = 0, \tag{4.22}$$

and the fifth equation gives that

$$(\alpha + 1)(\lambda\beta - 1) - \frac{1}{2}\beta^2 \left(1 + \frac{E(0)^2}{C(0)^2}\right) = 0. \tag{4.23}$$

Finally, the sixth equation gives that

$$\alpha^2 - \lambda\beta\alpha + \lambda\beta - 1 = 0. \tag{4.24}$$

Equation (4.24) can be immediately solved to find α in terms of λ and β

$$\alpha = \begin{cases} \lambda\beta - 1 \\ 1 \end{cases} \tag{4.25}$$

Equations (4.22) and (4.23) can be combined to eliminate the $E(0)^2/C(0)^2$ term, and after simplifying, this expression becomes

$$\beta^2 - \big[2\lambda(\alpha + 1) + \alpha - 1\big]\beta - 2\lambda\alpha(1 - \alpha) + 2(\alpha + 1) = 0, \tag{4.26}$$

which can be combined with equations (4.25) to solve for β as a function of λ. If $\alpha = \lambda\beta - 1$, then

$$\beta = \begin{cases} \dfrac{2}{\lambda - 1} \\[2ex] \dfrac{2\lambda}{2\lambda^2 - 1} \end{cases} \qquad (\alpha = \lambda\beta - 1). \tag{4.27}$$

If $\alpha = 1$, then the solutions are

$$\beta = 2\lambda \pm 2\sqrt{\lambda^2 - 1} \qquad (\alpha = 1). \tag{4.28}$$

The roots in equations (4.27) and (4.28) give four different combinations for α and β. After choosing α, the best choice for β depends on what one assumes about $C(0)$ and $E(0)$. If $\alpha = 1$ and $E(0) \neq 0$, then $\beta = 0$. This is not a desirable situation for an interior solution. If $\alpha = 1$ and $E(0) = 0$, then λ is a free parameter, and one can choose β to be either root in (4.28).

A better choice for α in the interior case might be $\alpha = \lambda\beta - 1$. This makes α, which appears in the metric, a function of the equation of state. As discussed in the previous section, one desires that $\lim_{\rho\to 0} C \propto \rho^{-\beta}$ to avoid singularities in the metric. Equations (4.27) and (4.28) show that in either choice for α, $\beta > 0$, which requires C to diverge as $\rho \to 0$. Suppose that $E(0)$ is finite and nonzero, then equation (4.22) becomes

$$(a - 1)(\beta - 2\lambda\alpha) = 0, \tag{4.29}$$

and equation (4.23) becomes

$$(\alpha + 1)(\lambda\beta - 1) - \frac{1}{2}\beta^2 = 0. \tag{4.30}$$

These equations are satisfied for all λ only if $\beta = 2\lambda/(2\lambda^2 - 1)$. However, equation (4.21) still must be satisfied, which is only possible if $\lambda = \frac{1}{2}(-1 \pm \sqrt{5})$, which gives that $\lambda \approx -1.62$ or $\lambda \approx 0.62$. This cannot be considered a possibility as it was shown earlier that $\lambda \geq 1$. Only if one assumes that $E(0) = 0$, can λ be chosen freely.

If one assumes that $\alpha = \lambda\beta - 1$ and that $E(0)$ diverges such that $E(0)/C(0)$ is finite, then equation (4.21) can only be satisfied for general λ if $\beta = 2/(\lambda - 1)$. If β is chosen to be $\beta = 2\lambda/(2\lambda^2 - 1)$, then again one must accept the unsatisfactory $\lambda = \frac{1}{2}(-1 \pm \sqrt{5})$.

While $\alpha = \lambda\beta - 1$ appears to be the best choice for α, how can one decide which scenario for $E(0)$ is appropriate so that a root for β can be chosen? The way out of this quandary is to use the elementary flatness condition in equation (4.14). This condition frequently requires that[57]

$$\lim_{\rho\to 0} Q = \text{finite}, \tag{4.31}$$

[57] Quevedo H 1990 *Fortschr. Phys.* **38** 733.

where Q is given by

$$Q = -\frac{\rho^{\alpha-\beta}}{(\alpha-\beta)}\frac{B}{\eta C^2}\left[\beta E + \eta(\eta^2+1)E'\right]. \tag{4.32}$$

Unfortunately, more must be known about the behavior of the metric components at $\rho = 0$ before this condition can be unambiguously applied. It appears that the best choice is a combination of $E(0)$ and β that leaves λ undetermined. This gives flexibility to any solution that might be found. For simplicity, this paper assumes that $E(0)/C(0)$ is finite, resulting in the following initial choices for α and β

$$\alpha = \beta + 1 = \lambda\beta - 1 = \frac{\lambda+1}{\lambda-1} \tag{4.33}$$

and

$$\beta = \frac{2}{\lambda - 1}. \tag{4.34}$$

As a side note, the above choice for α was made with the interior case in mind. A vacuum solution presents other possibilities, and in this case $\alpha = 1$ and $E(0) = 0$ may be the preferred choices.

4.3 The Power Solution

The field equations can be solved exactly by introducing the simplifying *Ansatz* that the metric components can be written as arbitrary powers of η. One makes the following assumptions

$$\begin{aligned} e^A &= \eta^s & B &= u\eta^w \\ C &= x\eta^c & E &= y\eta^e \end{aligned} \tag{4.35}$$

where c, e, s, u, w, x and y are all constants. Following the assumption of the previous section that $\lim_{\eta\to 0} E(0) \propto \lim_{\eta\to 0} C(0)$ requires that $e = c$. When the expressions in equation (4.35) are substituted into the field equations, the fourth equation becomes

$$w(w+1)\eta^2 + \alpha(\alpha-1) + w(w-1+2\alpha) - \frac{4\kappa\theta}{x^{2\lambda}}\eta^{s-2\lambda c} = 0 \tag{4.36}$$

Because each coefficient in this polynomial must equal zero, one notes immediately that unless

$$s - 2\lambda c = \begin{cases} 2 \\ 0 \end{cases} \tag{4.37}$$

then one is forced to set either $\kappa = 0$ or $\theta = 0$. If $\kappa = 0$, then only vacuum solutions are possible; if $\theta = 0$ then only dust solutions are possible. Thus one equation in (4.37) must be valid if one is to find an interior, perfect fluid solution.

With only the assumption that $e = c$, the field equations become the following set of ten equations. These equations are obtained by equating powers of η, and setting each coefficient of η equal to zero. These equations are:

$$c(w - c + 1) = 0 \tag{4.38}$$

$$b(a - b - 1 - w - 2) + c(a - 2) = 0 \tag{4.39}$$

$$(w + 1)(-2\lambda w + e) + \left(\frac{ye}{x}\right)^2 = 0 \tag{4.40}$$

$$\begin{aligned}(\alpha - 1)(c + b - 2\lambda a) + \left(\frac{y}{x}\right)^2 (b + e)^2 + w(b + c)& \\ - 2\lambda w(2\alpha + w - 1) = 0&\end{aligned} \tag{4.41}$$

$$w(1 + \alpha + w - \lambda\beta) - s(\alpha + 2w) + \frac{1}{2}c(b + c) + \frac{1}{2}\frac{ey^2}{x^2}(b + c) = 0 \tag{4.42}$$

$$w(w + 1)\eta^2 + \alpha(\alpha - 1) + w(w + 2\alpha - 1) - \frac{4\kappa\theta}{x^{2\lambda}}\eta^{s-2\lambda c} = 0 \tag{4.43}$$

$$(w - s)(w + 1) = 0 \tag{4.44}$$

$$\lambda\beta(\alpha + 1 + w) + (s - 1)(a + w + 1) - \frac{1}{2}(b + c)^2 - \frac{1}{2}\frac{y^2}{x^2}(b + c)^2 = 0 \tag{4.45}$$

$$s(w - 1) - \frac{1}{2}\left(c^2 + \frac{e^2 y^2}{x^2}\right) = 0 \tag{4.46}$$

$$\lambda\beta(1 - \alpha + w) + (w + \alpha)^2 + (s - 1) - s(w + 1) = 0. \tag{4.47}$$

A specific choice for s from equation (4.37) has not yet been made, and the fourth field equation still appears in the list above as a function of η, see equation (4.43). It can be shown that the only solution to these equations that satisfies the condition that $\lambda \geq 1$ is $c = w = s = e = 0$, which gives

$$\begin{aligned} A &= 0 \qquad & B &= u \\ C &= x \qquad & E &= y. \end{aligned} \tag{4.48}$$

where u is an arbitrary constant, and the ratio of the constants x and y is determined by the equation

$$\frac{y}{x} = \sqrt{\lambda^2 + \lambda - 1} \qquad \lambda \geq 1. \tag{4.49}$$

The parameter λ may be freely chosen as it is not determined by the field equations. The metric for this solution is

$$ds^2 = -x\rho^\beta \left(dt - \beta \frac{uy}{x^2} z\, d\phi\right)^2 + \frac{\rho^{\beta+2}}{x}(u^2\, d\phi^2 + d\rho^2 + dz^2). \tag{4.50}$$

This metric can be simplified by redefining the independent variables to absorb the extra constants. The variable transformation

$$\begin{aligned} t \to t' &= q^2 t \\ \phi \to \phi' &= \frac{u}{q}\phi \\ \rho \to \rho' &= \frac{\rho}{q} \\ z \to z' &= \frac{z}{q}, \end{aligned} \tag{4.51}$$

where

$$q = \sqrt[\beta+4]{x}, \tag{4.52}$$

gives the metric a more simple form:

$$ds^2 = -\rho^\beta \left(dt - \beta \frac{y}{x} z\, d\phi\right)^2 + \rho^{\beta+2}(d\phi^2 + d\rho^2 + dz^2). \tag{4.53}$$

The constants x and y are related by equation (4.49). The mass density in this solution is given by

$$\mu = \frac{\kappa}{\rho^\beta} \tag{4.54}$$

and the pressure is

$$p = \frac{\kappa\theta}{\rho^\beta}. \tag{4.55}$$

The product $\kappa\theta$ is determined by the relation

$$\kappa\theta = \tfrac{1}{4} x^{2\lambda} \alpha(\alpha - 1). \tag{4.56}$$

The singularities prevent the integration constant κ, normally determined by the boundary conditions, from being uniquely determined. The metric, the mass density μ, and the pressure p have obvious problems at $\rho = 0$. The constant coefficient of $d\phi^2$ suggests that ρ, ϕ, and z should be viewed as *rectangular* coordinates. The $dt\, d\phi$ term implies a linear change in time, but not a rotation. These considerations preclude any chance that this metric has any astrophysical application.

4.4 Other Possibilities

As mentioned previously, a complete solution of the reduced equations could not be found. The next step towards finding a solution is to introduce new assumptions. For example, one could assume a solution for a single function, and then attempt to solve the remaining equations. Upon examining the equations, one is tempted to choose a specific form for B, because the fourth field equation relates A and C to B and its derivatives. Thus, after choosing B, one immediately eliminates two unknowns from the equations. One attempt to simplify the equations did just this by assuming that $B = \eta^w$, where w is an arbitrary constant. However, when the field equations with this choice for B are examined for mathematical consistency (as in equations [4.15–17]), one finds that $w = 0$, which requires that $B =$ constant. If $B =$ constant then the fourth field equation gives this relation for e^A and C

$$e^{2A} = \frac{\alpha(\alpha - 1)}{4\kappa\theta} C^{2\lambda}. \tag{4.57}$$

The metric for this solution has the form

$$ds^2 = -\rho^\beta C(dt + Q\, d\phi)^2 + \frac{1}{\rho^\beta C}\left[\rho^{2\alpha} d\phi^2 + k\rho^{2(\lambda\beta - 1)} C^{2\lambda}(d\rho^2 + dz^2)\right], \tag{4.58}$$

where k is a constant. However, the behavior of this metric as $\rho \to 0$ is not encouraging. If, as mentioned earlier in equations (4.6–7), $\lim_{\rho\to 0} C \propto \rho^{-\beta}$, then the metric still as a singularity in the $g_{\rho\rho}$ and g_{zz} components. A quick analysis of the metric in (4.58) shows that a singularity is unavoidable if $B =$ constant.

One additional *Ansatz* that simplifies the equations is to require that E is a function of C. This simplification is related to a well known simplification of the vacuum field equations, which can be simplified by relating the real and complex parts of the Ernst potential. One particularly nice choice is to assume a linear relation between E and C, $E = kC$, where k is a constant. If this substitution is made, then equation (4.15) is integrable:

$$B^{2(\alpha-\lambda\beta)}C^{(1+k^2)\beta}e^{-2\alpha A} = \left(\frac{\eta^2}{\eta^2+1}\right)^{\tau}, \tag{4.59}$$

where

$$\tau = \alpha(\alpha-1) + \frac{1}{2}\beta^2(1+k^2) + 2\lambda\alpha\beta. \tag{4.60}$$

The implications of these *Ansätze* have not yet been explored, but they—most likely combined with additional simplifications—appear to be the most promising avenue for future research.

4.5 Conclusion

The field equations for the stationary, axially symmetric, rigidly rotating, relativistic perfect fluid were studied in this paper. The field equations were reduced to ordinary differential equations with a similarity transformation obtained from the field equations' Lie-Bäcklund symmetries. These equations could not be directly solved in their present form. A power series expansion of the field equations determined the two arbitrary parameters α and β. One interior solution could be found by assuming that the unknown functions are arbitrary powers of η. The metric for this solution suggests that the spacetime has cartesian symmetries. The mass density and pressure are singular along the rotation axis, and the metric degenerates to $ds^2 = 0$ on axis.

The ordinary differential equations probably will not be solvable without finding additional simplifications by making more assumptions. Only one assumption looks particularly promising at this time: the assumption that E is an arbitrary function of C. A linear relation between E and C particularly simplifies the equations.