THREE

# Symmetry Analysis

This chapter calculates the field equations' Lie-Bäcklund symmetries, and uses the equations' algebraic symmetries to solve for a similarity transformation. The transformation requires the metric functions to possess a special functional dependence, one that can be expressed with similarity variables, allowing the resulting field equations to be reduced to ordinary differential equations. Simple symmetries, such as invariance under the transformation $(\rho, z \to \rho, \rho/z)$, could be guessed by inspecting the equations. It was hoped that a full Lie-Bäcklund analysis would yield more complicated or subtle symmetries, involving perhaps special equations of state, allowing the equations to be greatly simplified for these cases. Alas, this was not the case, but that is skipping to the chapter's end. The Lie-Bäcklund symmetries are calculated with the Harrison-Estabrook method using differential forms, and the calculations were performed using the computer algebra program *Maple V* Release 3. The Harrison-Estabrook method is reviewed before solving for the symmetry generators.

## 3.1 The Harrison-Estabrook Method

The Harrison-Estabrook method for calculating Lie-Bäcklund symmetries or invariance groups of differential equations is a geometric approach based on Cartan's work with exterior differential forms. The following brief explanation of the Harrison-Estabrook procedure is based on their 1971 paper in the *Journal of Mathematical Physics* that contains a complete description of the method and provides several worked examples drawn from several fields in physics.[48] While Lie-Bäcklund symmetries can be calculated with standard tensor techniques,[49] the differential forms' skew-symmetric property often simplifies the calculations. In addition, exterior differential forms express the integration conditions naturally.

Briefly, one begins by writing the differential equations as a set of first order equations. The differential equations are then expressed in terms of an equivalent ideal of differential forms. This choice of forms is not unique, and one should choose a set of forms that simplifies the calculations. Any sets of differential equations that generate the same ideal must be algebraically equivalent. One then searches for all infinitesimal variable transformations that map the ideal into itself. Each such transformation may be represented by a vector field in the space of variables, which is called an *isovector* or symmetry generator. The method produces a usually over-determined set of partial differential equations for the isovectors' components. After using the Harrison-Estabrook method to find the symmetry generators, the similarity variables can be found by integrating the infinitesimal transformation equations to yield a finite transformation, which represents the Lie-Bäcklund symmetry. The similarity variables will be the integration "constants." The employment of similarity variables in this transformation reduces the field equations to ordinary differential equations. However, the total number of independent and dependent variables before and after the transformation remains the same. A more detailed explanation of the Harrison-Estabrook method follows below.

Cartan developed a formalism for expressing first order partial differential equations as a closed set of differential forms. A set of partial differential equations includes $s$ independent variables $x^A$ $(A = 1, \ldots, s)$, and $n - s$ dependent variables $u^i$ $(i = s + 1, \ldots, n)$. This set of differential equations is represented by

$$G(x^A, u^i, u^j_{,B}) = 0. \tag{3.1}$$

[48] Harrison B K and Estabrook F B 1971 *J Math. Phys.* **12** 653.

[49] Stephani H 1989 *Differential Equations: Their Solution Using Symmetries* (Cambridge: Cambridge University Press) pp. 232–245.

The differential forms equivalent to $G(x^A, u^i, u^i_{,B})$ exist in a $n$-dimensional manifold $\mathcal{M}$, and comprise a closed set of forms $R$,

$$R = \{\, w^\alpha; \quad \alpha = 1, \ldots, N \,\}. \tag{3.2}$$

The $\alpha$-index numbers the forms in the ideal, and the rank of $w^\alpha$ is denoted by $p_\alpha$. $R$ forms the basis for a differential ideal $\mathcal{I}$ of forms obeying a Grassmann algebra.

The closure condition for a set of forms is required to make the system completely integrable, and states that the exterior derivative of every form $w^\alpha$ in $\mathcal{I}$ be contained in $\mathcal{I}$

$$d\mathcal{I} \subset \mathcal{I}. \tag{3.3}$$

An unclosed set of forms may be closed by adding new forms to the ideal, and often these additional forms express the integration conditions.

The integral manifold is a submanifold of $\mathcal{M}$ obtained by first sectioning the forms and then annulling them. Annulling the forms simply means that the forms are set equal to zero. The sectioning operator transforms the original form $w$ into the sectioned form $\hat{w}$ (denoted with a hat) by the following prescription

$$du^{\hat{i}} = \frac{\partial u^i}{\partial x^A} dx^A \qquad dx^{\hat{A}} = dx^A. \tag{3.4}$$

Sectioning the forms expresses the dependence of $u^i$ on the independent variables $x^A$. A result of sectioning the forms is that the integral manifold is $s$-dimensional. The forms $dx^{\hat{A}}$ vary freely on the integral manifold, allowing the independent variables $x^A$ to be used as coordinates. The sectioned and annulled forms provide a functional specification of the other $n - s$ dependent variables in terms of the independent variables. Thus the integral manifold geometrically represents the partial differential equations' solution. The sectioned, annulled forms express their equivalent differential equations. Procedures exist for classifying the manifold $\mathcal{M}$ as a general or singular manifold. Differential forms that correspond to a well-posed set of differential equations are "in involution," and generate general manifolds. Usually, physically motivated equations form a well-posed set of equations. For this reason the involution criteria will not be discussed here.

A Lie-Bäcklund symmetry represents an invariance property of differential equations. In Cartan's geometric picture, a Lie-Bäcklund symmetry corresponds to any transformation in the space of the independent and dependent variables and their derivatives on the manifold $\mathcal{M}$ that leaves the ideal $\mathcal{I}$ invariant, resulting in an algebraically equivalent set of basis differential forms. Consider infinitesimal transformations of the dependent and independent variables generated by an $n$-dimensional vector field $\boldsymbol{V}$

$$x^A \rightarrow x^{\tilde{A}} = x^A + \epsilon V^A \tag{3.5}$$

$$u^i \rightarrow u^{\tilde{i}} = u^i + \epsilon V^i. \tag{3.6}$$

The vector field components are functions of all $n$—the dependent and independent—variables. The Lie derivative expresses any change the forms $R$ undergo through this "dragging" transformation. If $\boldsymbol{V}$ is a linear combination of all isovectors, then $\boldsymbol{V}$ is the generator of the equations' invariance group. The ideal's invariance under transformations along $\boldsymbol{V}$ is expressed mathematically as

$$\pounds_V \mathcal{I} \subset \mathcal{I}. \tag{3.7}$$

Thus the Lie derivative of any form $w^\alpha \in \mathcal{I}$ can be expressed as a linear combination of the forms in the ideal, which leads to the alternate expression

$$\pounds_V w^\alpha = \sum_\beta \lambda^\alpha_\beta \wedge w^\beta, \qquad (\mathrm{mod}\ \mathcal{I} = 0) \tag{3.8}$$

where $\lambda^\alpha_\beta$ is a multiplier of rank $p_\alpha - p_\beta$, and forms $w^\beta$ with $p_\beta > p_\alpha$ are neglected in the sum. The partial differential equations for the components of $\boldsymbol{V}$ are obtained by eliminating the $\lambda^\alpha_\beta$ multipliers from the system of equations (3.8) and annulling the forms by setting every coefficient in the expression to zero. An example of this process is included in Appendix B. The maximum number of possible resulting equations for the isovectors is[50]

$$\begin{array}{l}\text{Maximum}\\ \text{number of}\\ \text{equations}\end{array} = \tfrac{1}{2} N \big[s(n-s) - N\big]\big[s(n-s) - N + 2\big]. \tag{3.9}$$

[50] Carminati J, Devitt J S and Fee G J 1992 *J Symbolic Computation* **14** 103.

Fortunately this maximum is rarely achieved, and complicated systems are often not intractable. However these numbers are useful to determine the approximate workload required to solve for a system's isovector.

The infinitesimal transformation described above in (3.5–6) can be considered a "passive" point transformation or coordinate system transformation. But a passive transformation is equivalent to an "active" point transformation of the manifold's basis forms $R \to S$, where $S$ is a new basis of the differential ideal $\mathcal{I}$. The ideal's invariance means that the differential equations obtained by sectioning and annulling the forms $S$ are algebraically equivalent to the differential equations obtained from $R$. Thus isovector transformations can map solutions of differential equations into other solutions.

### 3.2 The Field Equations As Differential Forms

The stationary, axially symmetric, rigidly rotating, perfect fluid field equations were derived in Chapter 2, and were found to be:

$$\psi_{,\rho\rho} + \psi_{,zz} + \frac{1}{s}(s_{,\rho}\psi_{,\rho} + s_{,z}\psi_{,z}) - \frac{2}{f}(\psi_{,\rho}f_{,\rho} + \psi_{,z}f_{,z}) = 0 \tag{3.10}$$

$$\begin{aligned} f_{,\rho\rho} + f_{,zz} + \frac{1}{f}(\psi_{,\rho}^2 + \psi_{,z}^2 - f_{,\rho}^2 - f_{,z}^2) + \frac{1}{s}(s_{,\rho}f_{,\rho} + s_{,z}f_{,z}) \\ = 2e^{2\zeta}(\mu + 3p) \end{aligned} \tag{3.11}$$

$$\frac{1}{s}(s_{,z}\zeta_{,\rho} + s_{,\rho}\zeta_{,z} - s_{,\rho z}) = \frac{1}{2f^2}(f_{,\rho}f_{,z} + \psi_{,\rho}\psi_{,z}) \tag{3.12}$$

$$\frac{1}{s}(s_{,\rho\rho} + s_{,zz}) = \frac{4}{f}pe^{2\zeta} \tag{3.13}$$

$$\frac{1}{s}(\zeta_{,\rho}s_{,\rho} - \zeta_{,z}s_{,z} - s_{,\rho\rho}) - \zeta_{,\rho\rho} - \zeta_{,zz} = \frac{1}{2f^2}(f_{,\rho}^2 + \psi_{,\rho}^2) - \frac{4}{f}pe^{2\zeta} \tag{3.14}$$

$$\frac{1}{s}(\zeta_{,z}s_{,z} - \zeta_{,\rho}s_{,\rho} - s_{,zz}) - \zeta_{,\rho\rho} - \zeta_{,zz} = \frac{1}{2f^2}(f_{,z}^2 + \psi_{,z}^2) - \frac{4}{f}pe^{2\zeta}. \tag{3.15}$$

The first step in writing these field equations with differential forms is make them first order in all variables by introducing *jet* variables. For example, defining $a \equiv \psi_{,\rho}$ enables a first order expression for $\psi_{,\rho\rho}$: $\psi_{,\rho\rho} = a_{,\rho}$. The following jet variables are defined to write the field equations with differential forms

$$\begin{aligned} a &= \psi_{,\rho} & b &= \psi_{,z} \\ c &= f_{,\rho} & k &= f_{,z} \\ h &= s_{,\rho} & w &= s_{,z}. \end{aligned} \tag{3.16}$$

From the above list, one notes that no jet variables have been defined for the first derivatives of $\zeta$. Each jet variable is an additional dependent variable, adding an additional dimension to the manifold $\mathcal{M}$ and the isovector $\boldsymbol{V}$, further complicating the differential equations for its components obtained from (3.8). Taking the difference of (3.14) and (3.15) eliminates all second derivatives of $\zeta$; this difference has symmetry properties similar to the two original equations

$$2(\zeta_{,\rho}s_{,\rho} - \zeta_{,z}s_{,z}) - s_{,\rho\rho} + s_{,zz} = \frac{s}{2f^2}(f_{,\rho}^2 + \psi_{,\rho}^2 - f_{,z}^2 - \psi_{,z}^2). \tag{3.17}$$

Using (3.17) in lieu of (3.14) and (3.15) in the symmetry analysis simplifies the expressions for $\boldsymbol{V}$ by reducing the number of forms and jet variables, specifically: the number of forms from 15 to 14, the number of jet variables from 8 to 6, the number of two-form components from 91 to 66, reducing the number of potential isovector component equations from 1,462 to 840—a substantial savings in algebra!

Having defined jet variables to make the field equations first order and decided on a representative set of field equations to analyze, one now writes these equations as differential forms. If all forms in $R$ have the same rank, then the multipliers $\lambda^{\alpha}_{\beta}$ are constants (rank 0), greatly simplifying the multiplier elimination. To take advantage of this simplification, all forms describing the field equations will have rank 2. The jet variable equations in (3.16) expressed as second rank differential forms are

$$\begin{aligned} \alpha^1 &= -\,d\psi\,d\rho + b\,dz\,d\rho & \alpha^2 &= -\,d\psi\,dz + a\,d\rho\,dz \\ \alpha^3 &= -\,df\,d\rho + k\,dz\,d\rho & \alpha^4 &= -\,df\,dz + c\,d\rho\,dz \\ \alpha^5 &= -\,ds\,d\rho + w\,dz\,d\rho & \alpha^6 &= -\,ds\,dz + h\,d\rho\,dz, \end{aligned} \tag{3.18}$$

where a $\wedge$ product is understood between differentials. The field equations (3.10–13) and (3.17) in terms of forms are

$$\beta^1 = da\,dz - db\,d\rho + \left[\frac{1}{s}(ah + wb) - \frac{2}{f}(ac + bk)\right] d\rho\,dz \tag{3.19}$$

$$\begin{aligned}\beta^2 = {} & dc\,dz - dk\,d\rho + \left[\frac{1}{f}(a^2 + b^2 - c^2 - k^2)\right.\\ & \left. + \frac{1}{s}(hc + wk) - 2e^{2\zeta}(\mu + 3p)\right] d\rho\,dz\end{aligned} \tag{3.20}$$

$$\beta^3 = w\,d\zeta\,dz - h\,d\zeta\,d\rho + dh\,d\rho - \frac{s}{2f^2}(ab + ck)\,d\rho\,dz \tag{3.21}$$

$$\begin{aligned}\beta^4 = {} & 2h\,d\zeta\,dz + 2w\,d\zeta\,d\rho - dh\,dz - dw\,d\rho\\ & - \frac{s}{2f^2}(a^2 + c^2 - b^2 - k^2)\,d\rho\,dz\end{aligned} \tag{3.22}$$

$$\beta^5 = dh\,dz - dw\,d\rho - \frac{4sp}{f}e^{2\zeta}\,d\rho\,dz. \tag{3.23}$$

Finally, the following forms express the integration conditions for the jet variables, and are required to close the ideal

$$\begin{aligned}\gamma^1 &= da\,d\rho + db\,dz \qquad \gamma^2 = dc\,d\rho + dk\,dz\\ \gamma^3 &= dh\,d\rho + dw\,dz.\end{aligned} \tag{3.24}$$

The set $R = \{\,\alpha^1, \dots, \alpha^6, \beta^1, \dots, \beta^5, \gamma^1, \dots, \gamma^3\,\}$ is a closed ideal of differential forms equivalent to the original field equations. This equivalence can be verified by sectioning and annulling the forms in $R$ to reobtain the original equations. Closure can be verified by confirming that all exterior derivatives are contained in $R$. The Harrison-Estabrook procedure can now be implemented to produce the partial differential equations for the isovector components corresponding to any Lie-Bäcklund symmetries.

Equation (3.9) indicates that solving this system involves a maximum of 840 equations. While certainly better than facing 1,462 equations, this is still a large system to solve by hand. Luckily the computer algebra program *Maple V* can perform most of the calculations, as discussed in the next section.

## 3.3 Using Maple V

Carminati and Devitt have automated the Harrison-Estabrook method in the `liesymm` package written for the computer algebra program *Maple V*.[51] This package not only performs the Harrison-Estabrook procedure, but additionally attempts to simplify and sometimes even solve the resulting equations. The package has the additional advantage that its code can be viewed using *Maple's* `print` command. For example, the `determine` procedure's code can be viewed using the commands:

```
interface(verboseproc=2);
print(liesymm[determine]);
```

However, as with all research computer use, this package must be used with considerable caution.

The `liesymm` package automates the entire Harrison-Estabrook procedure in a single command: `determine`. The user enters the differential equations and specifies the independent and dependent variables. The `determine` command calls a series of primitive functions to generate the differential forms, ensure that the ideal is closed, take Lie derivatives of the forms, perform the modulo reduction of the forms and set their resulting coefficients equal to zero to produce the differential equations for the isovector components. The package's `autosimp` command then can be used to simplify and sometimes solve these equations for the isovector components. However, the equations resulting from the `determine` command could not be adequately reconciled with hand calculations for simple nonlinear differential equations, such as the Korteweg-de Vries equation

$$u_{,t} + uu_{,x} + \epsilon u_{,xxx} = 0, \tag{3.25}$$

and nonlinear heat equations with temperature dependent specific heat functions

$$u_{,t} = (e^u u_{,x})_{,x} \qquad u_{,t} = (u^a u_{,x})_{,x} \tag{3.26}$$

[51] Carminati J, Devitt J S and Fee G J 1992 *J Symbolic Computation* **14** 103. Heck A 1993 *Introduction to Maple* (New York: Springer Verlag) pp. 413–417. Char B W, Geddes K O, Gonnet G H, Leong B L, Monagan M B and Watt S M 1992 *Maple V Library Reference Manual* (New York: Springer Verlag) pp. 540–560.

where $a$ is an arbitrary constant. The difficulties with the Korteweg-de Vries equation are detailed in Appendix B.

The inability to reproduce `determine's` results, combined with a desire for greater flexibility in specifying dependent variables, led to developing original *Maple* programs using `liesymm's` primitive functions. These original *Maple* programs comprise a package known as `Dtools`. Appendix A contains a short explanation of the `Dtools` package, the procedures' code, and an example of their use. The `Dtools` programs are not automatic routines, but were designed to minimize the time required to perform the symmetry analysis of the field equations. This minimized time also includes software development. To save time the multiplier elimination process was not completely automated, but requires significant operator interaction. In retrospect, this process probably should have been automated, making the `Dtools` package essentially automatic and more useful in solving a wider range of problems.

The `liesymm` package allows the user to specify all variables in the equations as either independent or dependent variables. The package assumes that all dependent variables are functions of the same independent variables. As noted from the field equations' Bianchi identities (2.115), the perfect fluid's pressure and mass density can be expressed as functions of only the dependent variable $f$, $\mu = \mu(f)$ and $p = p(f)$. The `liesymm` package does not allow the user to make any such specification, and if $p$ and $\mu$ are declared to be dependent variables, `liesymm` assumes $\mu = \mu(\rho, z)$ and $p = p(\rho, z)$. Since the author found no way around this, $p$ and $\mu$ were initially entered as constants, and the resulting Lie derivatives were later corrected by hand. While the function $p(f)$ is unknown, the Bianchi identities give that

$$\frac{dp}{df} = -\frac{1}{2f}(\mu + 3p). \tag{3.27}$$

This additional information can be included in the *Maple* code after the `liesymm` package is no longer needed. The command `diff/` can be used to define derivatives. For convenience the derivative $d\mu/df$ was defined to be the function $A(f)$. The derivatives $dp/df$ and $d\mu/df = A(f)$ were defined with the commands

```
`diff/p` := proc(f,x) -1/(2*f)*(m(f)+p(f)) * diff(f,x) end:
`diff/mu` := proc(f,x) A(f) * diff(f,x) end:
```

The expressions are multiplied by `diff(f,x)` to ensure that *Maple* carries through the chain rule.

The `Dtools` package produces two sets of equations: one set with complicated expressions (multiple-term expressions), and another set of "trivial" expressions (single-term expressions), where one function or derivative equals zero. The trivial equations are expressions such as $V^t_{,w} = 0$, and are sorted separately to keep track of simple relationships. After `Dtools` produces these two sets of equations, they must be solved by hand.

### 3.4 Solving for the Symmetry Generator

The invariance group generator $\boldsymbol{V}$ was found using the *Maple* computer system. The *Maple* environment is first initialized with a small program called `grforms`. This program loads the `liesymm` and `Dtools` packages, defines all required variables and defines and annuls the set of forms $R$. The operator then calls the `Dlie` procedure with a single form $w^\alpha$ as its argument, where $w^\alpha \in R$. The `Dlie` procedure then performs the Lie derivative $\pounds_V w^\alpha$ and invokes the invariance condition by equating it with a linear sum of forms in the ideal

$$\pounds_V w^\alpha = \lambda^\alpha_\beta w^\beta. \tag{3.28}$$

Here the multipliers $\lambda^\alpha_\beta$ are constants because all forms in $R$ have rank 2. The forms are sectioned and annulled, giving differential equation expressions for the isovector components. `Dlie` then sorts these expressions and returns them in three different groups: (1) equations with multipliers, (2) multiple-term equations without multipliers, and (3) trivial equations. `Dtools` names the multiple-term equations without multipliers ES (Equation Set), and the trivial equations ZS (Zero Set), names that will be adopted here for convenience. The operator then must manually instruct *Maple* to eliminate the multipliers from the remaining equations, moving all equations into either ES or ZS. The `Dsub` routine assists in this tedious process. With `Dsub` the operator instructs *Maple* to solve one equation for a multiplier, and then insert that expression into a list of of specified equations. As the multipliers are eliminated, the equations are added to ES using the `Dadd` routine.

This process is then repeated for every form in $R$, resulting in fourteen ES sets and fourteen ZS sets. It often happens that the Lie derivative of a form will produce complicated expressions containing a certain term like $V^t_{,k}$, while the Lie derivative of another form gives that $V^t_{,k} = 0$. The `Dsimp` routine searches every equation in ES for terms contained in ZS. If any are found they are set equal to zero. `Dsimp` then looks for any trivial equations this process has created in ES, moves them to ZS and repeats the simplification process until no further simplifications are possible. While this paragraph describes the simplification process occurring only at the end for clarity, in practice the ES and ZS sets are combined together and simplified after performing the Lie derivative of every form. This keeps ES and ZS to manageable sizes, alleviating possible computer memory problems. At this stage the `Dtools` routines are finished, having produced the two sets of equations that now must be solved.

Performing this operation on the field equation forms yields 65 multiple-term equations (ES) (ranging from several equations with 2–3 terms to some equations with approximately 40–50 terms) and 32 trivial equations (ZS). These equations were slowly solved with classic predator technique by attacking the smallest and weakest among ES, finding new trivial equations, and then using the `Dsimp` procedure to insert these new simplifications back into ES.

After many iterations, the equations were reduced to around twenty that could be solved by hand, resulting in a solution for the equations' isovectors. This solution for $\boldsymbol{V}$, a linear sum of all isovectors, was then verified using two independent methods. First, by directly substituting $\boldsymbol{V}$ into the original 65 equations, using *Maple* to verify that each equation reduced to $0 = 0$, and secondly by directly verifying that $\boldsymbol{V}$ satisfies equation (3.8). The second analysis was performed with modified `Dtools` routines.

Before discussing the equations' invariance group, comments regarding one Lie-Bäcklund symmetry are in order. This single Lie-Bäcklund symmetry, represented by $c_2$ in equation (3.35) below, only exists if the mass density obeys the relationship

$$\frac{d\mu}{df} = -\frac{\mu}{2fp}(\mu + p). \tag{3.29}$$

Because the $c_2$ symmetry is the only symmetry that involves the metric functions $f$ and $\psi$, this paper assumes that the relation in (3.29) is valid. Equation (3.29) can be combined with the Bianchi identities (2.115) to solve explicitly for the equation of state

$$\frac{dp}{d\mu} = \frac{p}{\mu} \tag{3.30}$$

$$p = \theta\mu, \tag{3.31}$$

where $\theta$ is a constant. The speed of sound in a perfect fluid is given by $v_s = dp/d\mu$. Special relativity restricts $v_s \leq c$, requiring in the current unit system that $\theta \leq 1$. The requirement that $p$ be positive everywhere inside the fluid[52] restricts $\theta$ to the interval $0 < \theta \leq 1$. The equation of state in (3.31) can now be used to integrate the Bianchi identities. First, define two constants $\sigma$ and $\lambda$

$$\lambda \equiv \frac{1}{4}\left(\frac{1}{\theta} + 3\right) \qquad \sigma \equiv \frac{1}{2}\left(\frac{1}{\theta} + 1\right). \tag{3.32}$$

The constant $\sigma$ will be used immediately and $\lambda$ will appear later in the field equations and in the expression for $\boldsymbol{V}$. The integration of the Bianchi gives

$$\mu = \kappa f^{-\sigma} \tag{3.33}$$

$$p = \kappa\theta f^{-\sigma}, \tag{3.34}$$

where $\kappa$ is an integration constant. The condition in equation (3.29) has thus determined the equation of state, and allowed explicit expressions for $p(f)$ and $\mu(f)$ to be found.

The field equations' invariance group generator $\boldsymbol{V}$

$$\begin{aligned}\boldsymbol{V} = (c_2\lambda - c_3)\frac{\partial}{\partial\zeta} + c_2 f\frac{\partial}{\partial f} + c_1 s\frac{\partial}{\partial s} + (c_2\psi + c_6)\frac{\partial}{\partial\psi} \\ + (c_3\rho + c_4)\frac{\partial}{\partial\rho} + (c_3 z + c_5)\frac{\partial}{\partial z}.\end{aligned} \tag{3.35}$$

[52] This physical constraint for perfect fluids is discussed in § 1.2. Wahlquist argues that this restriction might be relaxed. Again, see the discussion in § 1.2.

is a linear combination of the isovectors representing each Lie-Bäcklund symmetry. A single isovector or Lie-Bäcklund symmetry is associated with each arbitrary constant $c_i$ in the expression for $\boldsymbol{V}$, and a given symmetry's expression is obtained by setting all other constants to zero. The symmetries corresponding to each constant are listed for convenience in Table I and will now be individually discussed. The symmetries corresponding to $c_4 \neq 0$, $c_5 \neq 0$ or $c_6 \neq 0$ are translations in $\rho$, $z$ and $\psi$ respectively. These symmetries expresses the fact that $\rho$, $z$ and $\psi$ do not explicitly appear in the field equations, leaving the field equations invariant under translations of these quantities.

The other three symmetries correspond to more interesting invariance transformations. $c_1 \neq 0$ represents a symmetry corresponding to a scale change of $s$. Indeed $s$ appears in the field equations only in combinations such as $s_{,\rho}/s$. Thus one easily recognizes the equations' invariance under scale changes in $s$. $c_2 \neq 0$ represents a symmetry corresponding to a scale change of $f$ and $\psi$ combined with a translation of $\zeta$. (A translation of $\zeta$ is equivalent to a scale change of the field equations because $\zeta$ appears only in the function $e^{\zeta}$.) Again, this invariance is recognized by inspecting the equations, as well as the reason the equation of state must be $p = \theta\mu$ for this symmetry to exist. Finally, $c_3 \neq 0$ represents a symmetry corresponding to a scale change of $\rho$ and $z$, again combined with a simultaneous translation of $\zeta$. This symmetry represents a scale change of the bisurface orthogonal to the spacetime's two Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. This is a scale change of the dimensions labeled by $\rho$ and $z$ only—the dimensions labeled by $t$ and $\phi$ are left unchanged.

### 3.5 Reduction to Ordinary Differential Equations

The previous section demonstrated that the field equations have six Lie-Bäcklund symmetries. The existence of these symmetries guarantees that a similarity transformation exists that reduces the number of independent variables. However, the precise form of this transformation is still unknown. Fortunately, Lie-Bäcklund symmetries provide a systematic approach to solving for the similarity transformation and finding the similarity variables. Indeed the Lie-Bäcklund symmetry analysis was performed for this reason. This

TABLE I. The Field Equations' Invariance Group.

| No. | $V^{\gamma}$ | $V^{f}$ | $V^{s}$ | $V^{\psi}$ | $V^{\rho}$ | $V^{z}$ | Type |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | $s$ | 0 | 0 | 0 | Scale change |
| 2[a] | $\lambda$ | $f$ | 0 | $\psi$ | 0 | 0 | Scale change |
| 3 | $-1$ | 0 | 0 | 0 | $\rho$ | $z$ | Scale change |
| 4 | 0 | 0 | 0 | 0 | 1 | 0 | Translation |
| 5 | 0 | 0 | 0 | 0 | 0 | 1 | Translation |
| 6 | 0 | 0 | 0 | 1 | 0 | 0 | Translation |

[a] This symmetry only exists if $p = \theta\mu$, where $0 < \theta \leq 1$.

section discusses the mechanics of solving for the similarity transformation, and reduces the field equations to ordinary differential equations with such a transformation.

In equation (3.1) a set of first order differential equations was represented by

$$G(x^A, u^i, u^j_{,B}) = 0. \tag{3.36}$$

where $A$ and $B$ range over the independent variables and $i$ and $j$ run over dependent variables. $\boldsymbol{X}$ is a symmetry operator of the differential equations if

$$\boldsymbol{X}G(x^A, u^i, u^j_{,B}) = 0, \qquad (\mathrm{mod}\ G = 0). \tag{3.37}$$

A similarity transformation seeks to take advantage of this symmetry by finding a transformation in the space of variables that transforms $\boldsymbol{X}$ to its normal form[53]

$$\boldsymbol{X} = \frac{\partial}{\partial\omega}. \tag{3.38}$$

Such a transformation is always possible, and requires the introduction of $n$ new similarity variables defined by

$$\boldsymbol{X}\omega = 1, \qquad \boldsymbol{X}\varphi^{\Omega} = 0. \qquad \Omega = 1, \ldots, n-1 \tag{3.39}$$

[53] Stephani H 1992 pp. 170–174.

From the variables $\varphi^\Omega$ one chooses $s-1$ variables to be independent, and the remaining variables are dependent. Sometimes physical considerations require that certain $\varphi^\Omega$ are independent, but otherwise the choice is not unique. Indeed the choice of the variables $\omega$ and $\varphi^\Omega$ is not unique.

One does not need to solve for $\omega$ to find a (non-unique) set of similarity variables $\varphi^\Omega$; they can be obtained directly from the finite Lie-Bäcklund symmetry transformations. The invariance generator $\boldsymbol{V}$ was found by considering infinitesimal transformations

$$\boldsymbol{x} \to \tilde{\boldsymbol{x}} = \boldsymbol{x} + \epsilon\, \boldsymbol{V}. \tag{3.40}$$

This expression can be simplified by writing $dx^\alpha = x^{\tilde{\alpha}} - x^\alpha$. Then equation (3.40) becomes

$$dx^\alpha = \epsilon V^\alpha. \tag{3.41}$$

The finite Lie-Bäcklund transformations can be found by integrating[54]

$$\frac{dx^\alpha}{V^\alpha} = \frac{dx^1}{V^1}, \tag{3.42}$$

where the superscript 1 represents a specific component. The similarity variables $\varphi^\Omega$ are the "integration" constants obtained when integrating equation (3.42).

The components of $\boldsymbol{V}$ can be read off from equation (3.35). The constants $c_i$ in the equations are arbitrary, and for $c_3 \neq 0$, the expressions for the components can be simplified by using the ratios $\alpha$ and $\beta$, defined by

$$\alpha = \frac{c_1}{c_3}, \qquad \beta = \frac{c_2}{c_3}. \tag{3.43}$$

The components of $\boldsymbol{V}$ are

$$V^\zeta = \beta\lambda - 1 \tag{3.44}$$
$$V^f = \beta f \tag{3.45}$$
$$V^s = \alpha s \tag{3.46}$$
$$V^\psi = \beta\psi \tag{3.47}$$
$$V^\rho = \rho \tag{3.48}$$
$$V^z = z. \tag{3.49}$$

[54] Stephani H 1992 p. 174

Here the coefficients $c_4$, $c_5$ and $c_6$ have been set equal to zero. Equation (3.42) will be integrated with respect to $\rho$ to obtain the Lie-Bäcklund transformations and corresponding similarity variables. For example, the transformation of $\zeta$ can be found by

$$\int \frac{d\rho}{V^\rho} = \int \frac{d\zeta}{V^\zeta} \tag{3.50}$$

$$\int \frac{d\rho}{\rho} = \int \frac{d\zeta}{\lambda\beta - 1}$$

$$\zeta = (\lambda\beta - 1)\ln\rho + A, \tag{3.51}$$

where $A$ is the integration constant. This process is repeated for the all other variables. Two more examples will be given here. First, the transformation of $s$:

$$\int \frac{ds}{V^s} = \int \frac{d\rho}{V^\rho} \tag{3.52}$$

$$\int \frac{ds}{\alpha s} = \int \frac{d\rho}{\rho}$$

$$\ln s = \alpha \ln\rho + \ln B$$

$$s = \rho^\alpha B. \tag{3.53}$$

Again, $B$ is the integration constant. In the final example, the $z$ transformation is obtained in the following manner:

$$\int \frac{dz}{V^z} = \int \frac{d\rho}{V^\rho} \tag{3.54}$$

$$\int \frac{dz}{z} = \int \frac{d\rho}{\rho}$$

$$\ln z = \ln\rho - \ln\eta \tag{3.55}$$

The $z$ transformation is of special interest because $z$ is an independent variable. The other variables—$f$, $s$, $\psi$ and $\zeta$—are all metric coefficients, and cannot be interpreted as dependent variables. Thus, the variable $\eta = \rho/z$

is a new independent variable, and the other similarity variables will be functions of $\eta$. The Lie-Bäcklund symmetry transformations are

$$\begin{aligned}
\zeta &= (\lambda\beta - 1)\ln\rho + A(\eta)\\
s &= \rho^{\alpha}B(\eta)\\
f &= \rho^{\beta}C(\eta)\\
\psi &= \rho^{\beta}E(\eta),
\end{aligned} \tag{3.56}$$

where $A$, $B$, $C$ and $E$ are dependent variables and unknown functions of $\eta$, and $\alpha$ and $\beta$ are arbitrary constants. The functions $A$, $B$, $C$ and $E$ must be defined over the domain $\eta \in (-\infty, +\infty)$ because $\eta \to \pm\infty$ as $z \to 0$.

The field equations expressed with these new variables are given below. A superscript prime denotes ordinary differentiation with respect to $\eta$, for example $A' = dA/d\eta$.

$$\begin{aligned}
&(-\beta^2 - \beta + \alpha\beta)E + \eta(\alpha + 2\eta^2)E' + \eta^2(\eta^2+1)E'' + \beta\eta\frac{B'}{B}E - 2\beta\eta\frac{C'}{C}E\\
&\quad + \eta^2(\eta^2+1)\frac{B'}{B}E' - 2\eta^2(\eta^2+1)\frac{C'}{C}E' = 0
\end{aligned} \tag{3.57}$$

$$\begin{aligned}
&\beta(\alpha-1)C + \eta(\alpha+2\eta^2)C' + \eta^2(\eta^2+1)C'' + \eta^2(\eta^2+1)\left(\frac{E'^2}{C} - \frac{C'^2}{C}\right)\\
&\quad + \beta^2\frac{E^2}{C} + 2\beta\eta\frac{EE'}{C} + \beta\eta\frac{B'}{B}C + \eta^2(\eta^2+1)\frac{B'C'}{B}\\
&\quad - 2e^{2A}\kappa C^{-\sigma}(1+3\theta) = 0
\end{aligned} \tag{3.58}$$

$$\begin{aligned}
&(2+\alpha-\lambda\beta)B' - (2\eta B' + \alpha B)A' + \eta B''\\
&\quad + \frac{1}{2}\beta B\left(\frac{C'}{C} + \frac{EE'}{C^2}\right) + \frac{1}{2}\eta B\left[\left(\frac{C'}{C}\right)^2 + \left(\frac{E'}{C}\right)^2\right] = 0
\end{aligned} \tag{3.59}$$

$$\alpha(\alpha-1) + 2\eta(\alpha+\eta^2)\frac{B'}{B} + \eta^2(\eta^2+1)\frac{B''}{B} = \frac{4}{C}\theta\kappa C^{-\sigma}e^{2A} \tag{3.60}$$

$$\begin{aligned}
&\lambda\beta(\alpha+1) - \alpha^2 - \frac{1}{2}\beta^2 - 1 + \eta(\alpha - 2\eta^2)A' - \eta^2(\eta^2-1)\frac{A'B'}{B} - \eta^2\frac{B''}{B}\\
&\quad - \eta^2(\eta^2+1)A'' + (\lambda\beta - 1 - 2\alpha)\eta\frac{B'}{B} - \beta\eta\frac{C'}{C} - \frac{1}{2}\beta^2\left(\frac{E}{C}\right)^2
\end{aligned}$$

$$- \beta\eta\frac{EE'}{C^2} - \frac{1}{2}\eta^2\left[\left(\frac{C'}{C}\right)^2 + \left(\frac{E'}{C}\right)^2\right] + \frac{4}{C}\theta\kappa C^{-\sigma}e^{2A} = 0 \tag{3.61}$$

$$\begin{aligned}
&\lambda\beta(1-\alpha) + \alpha - 1 + \eta^2(\eta^2-1)\frac{A'B'}{B} - \eta(\alpha+2\eta^2)A' - \eta^4\frac{B''}{B} \\
&\quad + (1-\lambda\beta-2\eta^2)\eta\frac{B'}{B} - \eta^2(\eta^2+1)A'' - \frac{1}{2}\eta^4\left[\left(\frac{C'}{C}\right)^2 + \left(\frac{E'}{C}\right)^2\right] \\
&\quad + \frac{4}{C}\theta\kappa C^{-\sigma}e^{2A} = 0.
\end{aligned} \tag{3.62}$$

The fourth equation can be used to eliminate $C^{-\sigma}$ terms, simplifying the second, fifth and sixth equations. The field equations now are

$$\begin{aligned}
&(-\beta^2 - \beta + \alpha\beta)E + \eta(\alpha+2\eta^2)E' + \eta^2(\eta^2+1)E'' + \beta\eta\frac{B'}{B}E - 2\beta\eta\frac{C'}{C}E \\
&\quad + \eta^2(\eta^2+1)\frac{B'}{B}E' - 2\eta^2(\eta^2+1)\frac{C'}{C}E' = 0
\end{aligned} \tag{3.63}$$

$$\begin{aligned}
&(\alpha-1)(\beta-2\lambda\alpha)C + \eta(\alpha+2\eta^2)C' + \eta^2(\eta^2+1)C'' + \beta^2\frac{E^2}{C} \\
&\quad + \eta^2(\eta^2+1)\left(\frac{E'^2}{C} - \frac{C'^2}{C}\right) + 2\beta\eta\frac{EE'}{C} + \left[\beta - 4\lambda(\alpha+\eta^2)\right]\eta\frac{B'}{B}C \\
&\quad + \eta^2(\eta^2+1)\frac{B'C'}{B} - 2\lambda\eta^2(\eta^2+1)\frac{B''}{B}C = 0
\end{aligned} \tag{3.64}$$

$$\begin{aligned}
&(2+\alpha-\lambda\beta)B' - (2\eta B' + \alpha B)A' + \eta B'' \\
&\quad + \frac{1}{2}\beta B\left(\frac{C'}{C} + \frac{EE'}{C^2}\right) + \frac{1}{2}\eta B\left[\left(\frac{C'}{C}\right)^2 + \left(\frac{E'}{C}\right)^2\right] = 0
\end{aligned} \tag{3.65}$$

$$\alpha(\alpha-1) + 2\eta(\alpha+\eta^2)\frac{B'}{B} + \eta^2(\eta^2+1)\frac{B''}{B} = \frac{4}{C}\theta\kappa C^{-\sigma}e^{2A} \tag{3.66}$$

$$\begin{aligned}
&\lambda\beta(\alpha+1) - \alpha - \frac{1}{2}\beta^2 - 1 + \eta(\alpha-2\eta^2)A' - \eta^2(\eta^2-1)\frac{A'B'}{B} + \eta^4\frac{B''}{B} \\
&\quad - \eta^2(\eta^2+1)A'' + (\lambda\beta - 1 + 2\eta^2)\eta\frac{B'}{B} - \beta\eta\frac{C'}{C} - \frac{1}{2}\beta^2\left(\frac{E}{C}\right)^2 \\
&\quad - \beta\eta\frac{EE'}{C^2} - \frac{1}{2}\eta^2\left[\left(\frac{C'}{C}\right)^2 + \left(\frac{E'}{C}\right)^2\right] = 0
\end{aligned} \tag{3.67}$$

$$\lambda\beta(1-\alpha)+\alpha^2-1+\eta^2(\eta^2-1)\frac{A'B'}{B}-\eta(\alpha+2\eta^2)A'+\eta^2\frac{B''}{B}$$
$$+(1-\lambda\beta+2\alpha)\eta\frac{B'}{B}-\eta^2(\eta^2+1)A''$$
$$-\frac{1}{2}\eta^4\left[\left(\frac{C'}{C}\right)^2+\left(\frac{E'}{C}\right)^2\right]=0. \tag{3.68}$$

Thus the field equations have been reduced to ordinary differential equations with a similarity transformation obtained from Lie-Bäcklund symmetries. Unfortunately, these equations are very complicated, fraught with a plethora of nonlinear terms. As mentioned in the introduction, it was hoped that some special cases would allow the field equations to be greatly simplified. However, the field equations' Lie-Bäcklund symmetries turned out to be quite simple, and several of them could have been easily guessed simply by inspecting the equations. Likewise, the new independent variable $\eta = \rho/z$ could have been guessed from the outset.

The field equations (3.63–68) are six equations containing four unknown metric functions. Because there are more equations than unknowns, the equations can either be redundant or over-determined. An analysis given in § 4.1 below combines all six field equations into a single equation that reduces to $0 = 0$. This result indicates that the equations are redundant rather than over-determined. One reason that the field equations are redundant is that they still must satisfy the Bianchi identities. A simple example of redundancy in the field equations is the Schwarzschild case. While the redundancy in the highly symmetric Schwarzschild case is easily seen, the complex and nonlinear nature of the field equations (3.63–68) masks their inherent redundancy.

### 3.6 A New Expression for Q

The new variable $\eta$ was introduced to reduce the field equations to ordinary differential equations. Although the function $Q$ does not appear directly in the field equations, its appearance in the metric compels one to also express $Q(\rho, z)$ in terms of the new variables, $Q(\rho, \eta)$. Because both functions are named $Q$, but contain different independent variables, the functional dependence will be made explicit when taking partial derivatives below. From (2.113–114), $Q(\rho, z)$ and $\psi$ are related by

$$Q(\rho, z)_{,\rho} = \frac{s}{f^2}\psi_{,z} \tag{3.69}$$

$$Q(\rho, z)_{,z} = -\frac{s}{f^2}\psi_{,\rho} \tag{3.70}$$

These equations can be written in terms of the similarity variables

$$Q(\rho, z)_{,\rho} = -\rho^{\alpha-\beta-1}\eta^2\frac{BE'}{C^2} \tag{3.71}$$

$$Q(\rho, z)_{,z} = -\rho^{\alpha-\beta-1}\frac{B}{C^2}(\beta E + \eta E'). \tag{3.72}$$

These derivatives of $Q(\rho, z)$ can be expressed alternatively in terms of $Q(\rho, \eta)$

$$Q(\rho, z)_{,\rho} = Q(\rho, \eta)_{,\rho} + \eta_{,\rho}Q(\rho, \eta)_{,\eta} \tag{3.73}$$

$$= Q(\rho, \eta)_{\rho} + \frac{\eta}{\rho}Q(\rho, \eta)_{,\eta} \tag{3.74}$$

$$Q(\rho, z)_{,z} = \eta_{,z}Q(\rho, \eta)_{,\eta} \tag{3.75}$$

$$= -\frac{\eta^2}{\rho}Q(\rho, \eta)_{,\eta} \tag{3.76}$$

Equations (3.71–72) and (3.74,76) can be used to solve for the derivatives of $Q(\rho, \eta)$ in terms of $Q(\rho, z)$

$$Q(\rho, \eta)_{,\rho} = Q(\rho, z)_{,\rho} + \frac{1}{\eta}Q(\rho, z)_{,z} \tag{3.77}$$

$$Q(\rho, \eta)_{,\eta} = -\frac{\rho}{\eta^2}Q(\rho, z)_{,z} \tag{3.78}$$

These expressions can now be written in terms of the new variables

$$Q(\rho, \eta)_{,\rho} = \rho^{\alpha-\beta-1}\frac{B}{\eta C^2}\left[\eta(\eta^2 + 1)E' + \beta E\right] \tag{3.79}$$

$$Q(\rho, \eta)_{,\eta} = \rho^{\alpha-\beta-1}\frac{B}{\eta^2 C^2}(\beta E + \eta E') \tag{3.80}$$

Equation (3.79) can be integrated to give

$$Q(\rho, \eta) = -\frac{\rho^{\alpha-\beta}}{(\alpha-\beta)}\frac{B}{\eta C^2}\left[\beta E + \eta(\eta^2 + 1)E'\right] + H(\eta), \tag{3.81}$$

where $H(\eta)$ is an arbitrary function of $\eta$ that can be determined by taking the derivative with respect to $\eta$ of equation (3.81), and equating it with (3.80). The first field equation (3.63) can be used to eliminate the $E''$ term that appears in the $\eta$-derivative of (3.81), and it can be shown that $H(\eta)$ is a constant, which will be taken to be zero: $H(\eta) = 0$. (The calculation is simple, yet the equations are cumbersome; hence the details will not given here.) Thus, $Q$ can be expressed

$$Q(\rho, \eta) = -\frac{\rho^{\alpha-\beta}}{(\alpha-\beta)} \frac{B}{\eta C^2} \left[\beta E + \eta(\eta^2+1)E'\right]. \tag{3.82}$$

This chapter used the field equations' Lie-Bäcklund symmetries to find a variable transformation that reduced the field equations to ordinary differential equations. The solution of these reduced equations is a similarity solution, and is a special case of the general solution. The similarity solution could be physically important, if the solution satisfies the appropriate boundary conditions. This latter point is still an open question, and can only be answered when more is known of the similarity solution. Unfortunately a complete solution of even the reduced equations could not be found. Some specialized solutions were investigated, as reported in the next chapter.