TWO

The Field Equations

Einstein's field equations equate mass-energy, represented by the stress-energy tensor $\boldsymbol{T}$, with the curvature of spacetime, as expressed by the Ricci tensor $\boldsymbol{R}$. Units are chosen such that $c = 1$ and $G = 1/4\pi$, and equation (1.1) is expressed in the equivalent form:

$$\boldsymbol{R} = 2\,\boldsymbol{T} - T\boldsymbol{g}. \tag{2.1}$$

This chapter derives the field equations for a uniformly rotating, rigid perfect fluid. But before delving into the field equations, one must choose a model for the matter distribution and an appropriate coordinate system.

2.1 Perfect Fluids

A physicist studying astrophysical objects has the choice of two simple materials when modeling the object's interior: dust and perfect fluids. Dust, as the name implies, is appropriate for very diffuse matter, where the intermolecular forces (pressures and stresses) are vanishingly small. The stress-energy tensor for uncharged dust

$$\boldsymbol{T} = \mu\, \boldsymbol{u} \otimes \boldsymbol{u} \tag{2.2}$$

contains only two important quantities: μ, the dust's mass density, and $\boldsymbol{u}$, the dust particles' four-velocity. Obviously dust is not an appropriate choice to model the congealed matter distributions of relativistic stars, where intermolecular forces become quite large. These intermolecular forces (or internal stresses and pressures) in fact contribute to the gravitational field. A simple model that includes pressure is the perfect fluid. A perfect fluid is characterized by its mass density μ, the four-velocity of its constituents $\boldsymbol{u}$ and an isotropic pressure p. The quantities μ and p are measured in the fluid's rest frame. The perfect fluid stress-energy tensor is

$$\boldsymbol{T} = (\mu + p)\boldsymbol{u} \otimes \boldsymbol{u} + p\boldsymbol{g}. \tag{2.3}$$

A perfect fluid has neither shear stresses, anisotropic pressures, nor viscosity and heat transfer. Perfect fluids are the fluids of choice for simple models of stellar objects: they account for the effects of pressure, while keeping all other aspects as simple as possible.

2.2 Coordinate Considerations

Having chosen a suitable mass distribution model, the spacetime's general properties require consideration. The metric tensor $\boldsymbol{g}$ determines the spacetime's local properties. At a fundamental level, the metric describes how coordinates determine physically meaningful quantities such as intervals, proper distances and proper times between two events. But the metric does much more: it determines the manifold's local topology and differential geometry.[32] Finding a solution to Einstein's field equations often begins by assuming an arbitrary metric and then solving the field equations to determine the metric components. The form of the assumed metric plays a large role in determining whether the equations can be solved.

The metric tensor is a second rank, symmetric tensor, and in a general spacetime the metric has ten independent components. As mentioned before, the Einstein field equations must be written in a specific coordinate system before a solution can be found, and for a general perfect fluid they are ten coupled, nonlinear, partial differential equations containing the ten metric components, and the perfect fluid's mass density, four-velocity and isotropic pressure. In their covariant expression the field equations admit a general coordinate transformation, and therefore determine only six of the ten metric components. Choosing a specific coordinate system allows one to impose four additional conditions on the metric components.

[32] The local nature of the Einstein field equations does not fix the spacetime's global topology.

The following shows how physical symmetries can be used to simplify the metric, and how the coordinate system can be adapted to the spacetime's symmetries. The treatment is cast in a covariant form for generality. Consider defining symmetries in terms of infinitesimal transformations generated by a vector $\boldsymbol{\xi}$:

$$x^{\alpha} \to x^{\tilde{\alpha}} = x^{\alpha} + \epsilon\, \xi^{\alpha}, \tag{2.4}$$

where ϵ characterizes the transformation's infinitesimal nature, $\epsilon \ll 1$. Infinitesimal transformations can represent any combination of rotations and translations, but not inversions. The Lie derivative describes quantitatively how geometric objects change under these "dragging" transformations. If the Lie derivative of a tensor $\boldsymbol{B}$ along $\boldsymbol{\xi}$ is equal to zero

$$\pounds_{\xi} \boldsymbol{B} = 0, \tag{2.5}$$

then $\boldsymbol{B}$ is unchanged along $\boldsymbol{\xi}$, and $\boldsymbol{\xi}$ represents a symmetry transformation of $\boldsymbol{B}$. In particular, spacetime symmetries correspond to transformations $\boldsymbol{x} \to \boldsymbol{x} + \epsilon\, \boldsymbol{\xi}$ that leave all of the spacetime's characteristics unchanged, as indicated in Killing's equation

$$\pounds_{\xi}\, \boldsymbol{g} = 0. \tag{2.6}$$

This special symmetry of $\boldsymbol{g}$ is called an *isometry*. The metric components of a spacetime with an isometry are independent of one coordinate y relative to some coordinate basis $\mathbf{d}x^{\alpha}$

$$\frac{\partial}{\partial y} g_{\mu\nu} = 0. \tag{2.7}$$

The coordinate y is a *cyclic* coordinate, analogous to its namesake in classical mechanics. A transformation generator $\boldsymbol{\xi}$ that satisfies (2.6) is a Killing vector. Killing vectors thus provide symmetries with the necessary covariant expression.

Einstein's field equations relate matter to geometry, consequently the spacetime and matter distribution must possess the same symmetries. The spacetime of an isolated, uniformly rotating star must also be stationary and axially symmetric in both interior and exterior regions. The spacetime contains a timelike Killing vector $\boldsymbol{\xi}$ and a spacelike Killing vector $\boldsymbol{\eta}$ with compact orbits. In a general theorem, Carter has shown that one may assume that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ commute everywhere.[33] A result of differential geometry states that if $[\boldsymbol{\xi},\boldsymbol{\eta}] = 0$, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ can define coordinate directions, and can be written as $\boldsymbol{\xi} = \partial/\partial x^0$ and $\boldsymbol{\eta} = \partial/\partial x^1$. Let t label the timelike coordinate and ϕ label the spacelike coordinate: $t = x^0$ and $\phi = x^1$. Choosing $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ to define two coordinate directions not only adapts the coordinate system to the symmetries, but also imposes two of the four coordinate conditions on $\boldsymbol{g}$. These constraints express metric isometries

$$g_{\mu\nu,t} = 0 \qquad g_{\mu\nu,\phi} = 0. \tag{2.8}$$

Papapetrou investigated families of 2-dimensional surfaces orthogonal to the surfaces of transitivity, obtained by varying (t,ϕ) while holding all other coordinates constant, and proved their existence for the vacuum case.[34] Carter extended Papapetrou's work and showed that these orthogonal surfaces exist in special non-vacuum cases, provided that the Ricci tensor is invertible with respect to the isometry group[35]:

$$\xi^\delta R_{\delta[\alpha}\xi_\beta\eta_{\gamma]} = 0 \qquad \eta^\delta R_{\delta[\alpha}\xi_\beta\eta_{\gamma]} = 0. \tag{2.9}$$

The Ricci tensor is invertible for spacetimes containing perfect fluids and satisfies (2.9) when the fluid's four-velocity obeys the circularity condition[36]

$$u_{[\alpha}\xi_\beta\eta_{\gamma]} = 0, \tag{2.10}$$

which requires the perfect fluid to "flow" in the plane spanned by $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$

$$\boldsymbol{u} = (u^t, u^\phi, 0, 0). \tag{2.11}$$

---

[33] Carter B 1970 *Commun. Math. Phys.* **17** 233.

[34] Papapetrou A 1966 *Ann. Inst. H. Poincaré* A iv 83.

[35] Carter B 1969 *J Math. Phys.* **10** 70.

[36] Kramer D, Stephani H, MacCallum M and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press) p. 193.

As one expects, a uniformly rotating perfect fluid conforms to these conditions, and thus Carter's orthogonal transitivity assures the existence of commuting vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\chi}$ that are orthogonal to—and also commute with—$\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Again one can introduce coordinates adapted to these vectors: $\boldsymbol{\zeta} = \partial/\partial x^2$ and $\boldsymbol{\chi} = \partial/\partial x^3$. The orthogonal relationship between the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ and the coordinate vectors $\boldsymbol{\zeta}$ and $\boldsymbol{\chi}$

$$\begin{aligned} g_{\mu\nu}\zeta^\mu\xi^\nu &= 0 \qquad & g_{\mu\nu}\zeta^\mu\eta^\nu &= 0 \\ g_{\mu\nu}\chi^\mu\xi^\nu &= 0 \qquad & g_{\mu\nu}\chi^\mu\eta^\nu &= 0 \end{aligned} \tag{2.12}$$

provides four further conditions on the metric tensor

$$g_{tA} = 0 \qquad g_{\phi A} = 0. \qquad\qquad A = 2, 3 \tag{2.13}$$

The general metric now takes the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt\, d\phi + g_{\phi\phi}d\phi^2 + g_{AB}dx^A\, dx^B, \tag{2.14}$$

where, following the condition in equation (2.8), the metric components are functions of x^A only, where $A = 2, 3$.

Invariance properties provide a quick and intuitive check that eq. (2.14) is the correct metric form, and further distinguish *static* and *stationary* spacetimes. The metric components of static and stationary spacetimes contain no explicit time dependence: $g_{\mu\nu,t} = 0$. A static spacetime is invariant under time inversion ($t \to -t$). Spacetime invariance under this inversion requires that $g_{tk} = 0$, in addition to the condition $g_{\mu\nu,t} = 0$. A stationary spacetime, on the other hand, is not invariant under simple time inversion, but is invariant under the transformation ($t, \phi \to -t, -\phi$). From invariance considerations one concludes in general that $g_{t\phi} \neq 0$ in stationary, rotating spacetimes, while the other off-diagonal terms g_{tA} are zero, as expressed in (2.13) above.

One further simplification results in no loss of generality. Consider the bisurface orthogonal to $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$

$$d\sigma^2 = g_{AB}dx^A\, dx^B. \tag{2.15}$$

A conformal mapping between two spacetimes $\boldsymbol{g}$ and $\tilde{\boldsymbol{g}}$ maintains the following relation

$$\begin{aligned} g_{\tilde{\mu}\tilde{\nu}} &= \Lambda g_{\mu\nu} \\ d\tilde{s}^2 &= \Lambda ds^2, \end{aligned} \tag{2.16}$$

where Λ is the conformal factor, a positive-definite function of x^A. Conformal mappings leave the Weyl conformal tensor $C^{\alpha}{}_{\beta\gamma\delta}$ invariant under the mapping, $\tilde{\boldsymbol{C}} = \boldsymbol{C}$. In two dimensions $C^{\alpha}{}_{\beta\gamma\delta}$ is identically zero, and conformal mappings of the form (2.16) are possible on the bisurface. Further, as $\boldsymbol{C} = 0$ the space is conformally flat, and there must exist a mapping

$$x^{\tilde{A}} = x^{\tilde{A}}(x^B) \tag{2.17}$$

such that isotropic coordinates $(x^{\tilde{2}}, x^{\tilde{3}})$ can be introduced.[37] After dropping the tildes, equation (2.15) becomes

$$d\sigma^2 = e^{\nu}\left[(dx^2)^2 + (dx^3)^2\right]. \tag{2.18}$$

Thus, without loss of generality the stationary, the axially symmetric spacetime metric may be written

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dt\, d\phi + g_{\phi\phi}d\phi^2 + e^{\nu}\left[(dx^2)^2 + (dx^3)^2\right], \tag{2.19}$$

with four unknown functions of x^A: g_{tt}, $g_{t\phi}$, $g_{\phi\phi}$ and ν.

Papapetrou has shown that the field equations assume a very simple form when the metric (2.19) is expressed in a slightly different, yet equivalent form[38]

$$ds^2 = -f(dt - Qd\phi)^2 + \frac{1}{f}\left[s^2d\phi^2 + e^{2\zeta}(d\rho^2 + dz^2)\right]. \tag{2.20}$$

Here "canonical" cylindrical coordinate labels (ρ, z) replace x^2 and x^3, and the metric functions f, Q, s and ζ are arbitrary functions of ρ and z. This is the metric used in this paper.

[37] Carmeli M 1982 *Classical Fields: General Relativity and Gauge Theory* (New York: John Wiley & Sons) pp. 72–76, 367. See also Islam J N 1985 *Rotating Fields in General Relativity* (Cambridge: Cambridge University Press) Chapter One.

[38] Carmeli M 1982 pp. 368–371.

Q is the metric's only off-diagonal metric component, and its physical significance deserves comment. In an asymptotically flat region free of gravitational waves, the exterior spacetime of a rotating body is described by the metric[39]

$$ds^2 = -\left[1 - \frac{2M}{r} + \frac{2M^2}{r^2} + O\left(\frac{1}{r^3}\right)\right] dt^2 - \left[4\epsilon_{jkl} S^k \frac{x^l}{r^3} + O\left(\frac{1}{r^3}\right)\right] dt\, dx^j$$
$$+ \left[\left(1 + \frac{2M}{r}\right)\delta_{jk} + O\left(\frac{1}{r^2}\right)\right] dx^j\, dx^k, \tag{2.21}$$

where S^k are components of the source's intrinsic angular momentum vector, and M is the total mass energy. In special relativity mass and angular momentum are defined with flux integrals, which require Minkowski-like coordinates for any consistent application. Near a strongly relativistic source where asymptotic coordinates do not exist, flux integrals—and thus the mass and angular momentum definitions—become meaningless. Thus Q is related to the source's angular momentum, but the precise connection can only be made in the weak field limit, the only region where angular momentum can even be defined. This connection also relates to the earlier comments regarding the spacetime's invariance properties. If $Q = 0$, then (2.20) in invariant under the inversion $(t \to -t)$, corresponding to a static spacetime as expected.

2.3 Rigid Rotation and Comoving Coordinates

Thus far only two coordinate conditions have been imposed on the metric functions, namely defining the coordinates (t, ϕ) in terms of the Killing vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. As the stress-energy tensor contains the perfect fluid four-velocity $\boldsymbol{u}$ (see equation [2.3] above), the final two coordinate conditions will be imposed by choosing comoving coordinates to further simplify the stress-energy tensor expressions.

---

[39] Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (New York: W. H. Freeman and Company) pp. 449, 452. Here the origin is placed at the system's center of mass and the coordinate system is the rest frame.

Born's constraint for rigid motion in special relativity requires that the body's rate-of-strain tensor $\boldsymbol{S}$ vanishes, a condition that transfers easily into general relativity via the strong equivalence principle. As the body's shear and expansion consequently also vanish, the world lines of the perfect fluid's constituent particles are a congruence of timelike, non-intersecting curves with constant orthogonal separation.[40] These trajectories can be adopted as new timelike coordinates for local observers, and constitute a comoving coordinate system.[41]

In comoving coordinates the spatial components of the fluid's four-velocity are zero. The final two coordinate conditions are applied as follows. The fluid's four-velocity obeys condition (2.10) that $\boldsymbol{u} = (u^t, u^\phi, 0, 0)$. One can write $u^\phi = \Omega u^t$, where rigid rotation requires the locally measured angular velocity Ω to be constant. The transformation $\phi \rightarrow \tilde{\phi} = \phi - \Omega t$ imposes the remaining conditions and shows that $u^{\tilde{\phi}} = 0$. The fluid's four-velocity in this comoving coordinate system can then be expressed with metric functions by recalling that $d\tau = -ds$

$$u^\alpha = -\frac{dx^\alpha}{ds} = \left(\frac{1}{\sqrt{f}}, 0, 0, 0\right). \tag{2.22}$$

This simple expression for $\boldsymbol{u}$ in terms of the metric function f justifies choosing comoving coordinates. It is worth emphasizing that this simple expression for $\boldsymbol{u}$ is only possible because the rotation is assumed to be rigid. This simplification would not be possible for a differentially rotating system.

2.4 The Einstein Field Equations

Calculating the field equations for the metric is now a straightforward process. However, Harrison developed a formalism that expresses the field equations for a general metric with a single isometry in an elegant form, and this formalism will be followed in deriving the field equations.[42] While

[40] Synge J L 1960 *Relativity: The General Theory* (Amsterdam: North-Holland Publishing Co.) pp. 114–117. See also Wahlquist H D and Estabrook F B 1966 *J Math. Phys.* **7** 894.

[41] Carmeli M 1982 pp. 190–192.

[42] Harrison B K 1968 *J Math. Phys.* **9** 1744.

the metric and resulting field equations are more general than required, a simple change of variables yields the field equations for the metric in equation (2.20). In adapting Harrison's procedure to metric (2.20), a few notation changes ease the way. Harrison uses k, which does not sum, to label a cyclic coordinate corresponding to a metric isometry. In the following t replaces k. Adopting this change, and setting the inconsequential scale factor $a = 1$, Harrison's metric is

$$ds^2 = -e^{2U}(dt + f_i dx^i)^2 - e^{-2U}\gamma_{ij} dx^i\, dx^j. \tag{2.23}$$

Here, as a reminder, the Latin indices represent the spatial coordinates only, except for t (and later z) which are coordinate labels, and all repeated indices are summed.

The three dimensional spatial metric γ_{ij} is left unspecified for generality. The field equations are simplified by defining geometric quantities derived from $\boldsymbol{\gamma}$. Specifically, γ_{ij} has an inverse defined by

$$\gamma^{ij}\gamma_{jk} = \delta^i{}_k. \tag{2.24}$$

The function γ is the determinant of γ_{ij}

$$\gamma = \det(\gamma_{ij}). \tag{2.25}$$

The Christoffel symbols and Ricci tensor derived from $\boldsymbol{\gamma}$ are denoted $\Sigma^i{}_{jk}$ and P_{ij} respectively, and are calculated in the standard way. The differential parameters of first and second order defined in terms of $\boldsymbol{\gamma}$ are[43]

$$\Delta_1(F) = \gamma^{ij} F_{,i} F_{,j} \tag{2.26}$$

$$\Delta_1(F, G) = \gamma^{ij} F_{,i} G_{,j} \tag{2.27}$$

$$\Delta_2(F) = \gamma^{ij} F_{|ij} = \gamma^{ij}(F_{,ij} - \Sigma^k{}_{ij} F_{,k}). \tag{2.28}$$

A subscript vertical stroke '|' represents covariant differentiation with respect to $\boldsymbol{\gamma}$, analogous to the semicolon convention. Finally, the antisymmetric quantity h_{ij} is defined by

$$h_{ij} = f_{i,j} - f_{j,i}. \tag{2.29}$$

[43] Eisenhart L P 1949 *Riemannian Geometry* (Princeton: Princeton University Press) p. 41.

Before calculating the Ricci tensor for the field equations, the metric and Christoffel symbols must first be found. The metric and inverse metric components are

$$\begin{aligned} g_{tt} &= -e^{2U} & g^{tt} &= -e^{-2U} - f_i f_j e^{2U}\gamma^{ij} \\ g_{ti} &= -f_i e^{2U} & g^{ti} &= f_j e^{2U}\gamma^{ij} \\ g_{ij} &= -f_i f_j e^{2U} - e^{-2U}\gamma_{ij} & g^{ij} &= -e^{2U}\gamma^{ij}. \end{aligned} \tag{2.30}$$

The Christoffel symbols for this metric are then

$$\Gamma^t{}_{tt} = e^{4U} f_i U_{,j}\gamma^{ij} \tag{2.31}$$

$$\Gamma^t{}_{ta} = U_{,a} + \tfrac{1}{2}e^{4U} f_i \gamma^{ib}(h_{ab} + 2f_a U_{,b}) \tag{2.32}$$

$$\Gamma^a{}_{tt} = -e^{4U} U_{,c}\gamma^{ac} \tag{2.33}$$

$$\Gamma^a{}_{bt} = \tfrac{1}{2}e^{4U}\gamma^{ac}(h_{cb} - 2f_b U_{,c}) \tag{2.34}$$

$$\begin{aligned} \Gamma^t{}_{ab} = {}& \tfrac{1}{2}(f_{a,b} + f_{b,a}) + 2(f_a U_{,b} + f_b U_{,a}) - f_c U_{,c}\gamma^{dc}\gamma_{ab} - f_c \Sigma^c{}_{ab} \\ & - \tfrac{1}{2}e^{4U} f_c \gamma^{dc}(f_b h_{da} + f_a h_{db}) + e^{4U} f_c \gamma^{dc} f_a f_b U_{,d} \end{aligned} \tag{2.35}$$

$$\begin{aligned} \Gamma^a{}_{bc} = {}& \tfrac{1}{2}e^{4U}\gamma^{ad}(f_b h_{dc} + f_c h_{db}) - e^{4U}\gamma^{ad} f_b f_c U_{,d} \\ & - U_{,c}\delta^a{}_b - U_{,b}\delta^a{}_c + U_{,d}\gamma^{ad}\gamma_{bc} + \Sigma^a{}_{bc}. \end{aligned} \tag{2.36}$$

The four dimensional Ricci tensor components are

$$R_{tt} = -e^{4U}\Delta_2(U) + \tfrac{1}{2}e^{8U}\gamma^{ab}\gamma^{ij}h_{ai}h_{bj} \tag{2.37}$$

$$R_{ta} = f_a R_{tt} + \tfrac{1}{2}e^{4U}\gamma^{ij}(-h_{aj,i} + \Sigma^e{}_{ij}h_{ae} + \Sigma^e{}_{aj}h_{ei} + 4h_{ia}U_{,j}) \tag{2.38}$$

$$\begin{aligned} R_{ab} = {}& f_a R_{tb} + f_b R_{ta} - f_a f_b R_{tt} + P_{ab} - 2U_{,a}U_{,b} + \gamma_{ab}\Delta_2(U) \\ & - \tfrac{1}{2}e^{4U}\gamma^{ij}h_{ai}h_{bj}. \end{aligned} \tag{2.39}$$

Equation (2.38) contains the term $-h_{aj,i}$. The integration condition requires the equality of mixed partial derivatives and gives an expression relating derivatives of h_{ab}

$$-h_{aj,i} = h_{ia,j} + h_{ji,a}, \tag{2.40}$$

allowing one to simplify (2.38) above to

$$R_{ta} = f_a R_{tt} + \tfrac{1}{2}e^{4U}\gamma^{ij}(h_{ia|j} + 4h_{ia}U_{,j}). \tag{2.41}$$

The stress energy tensor components are quite easily found using Equation (2.3), the stress-energy tensor for a perfect fluid. Combining the fluid four-velocity in comoving coordinates, as found in (2.22), with (2.3) gives the stress-energy tensor components to be

$$T_{tt} = \mu e^{2U} \tag{2.42}$$

$$T_{ta} = \mu e^{2U} f_a \tag{2.43}$$

$$T_{ab} = \mu e^{2U} f_a f_b - p e^{-2U} \gamma_{ab}. \tag{2.44}$$

The stress energy tensor trace T is

$$T = T^{\alpha}{}_{\alpha} = -\mu + 3p. \tag{2.45}$$

To simplify the field equations the gravitational field source tensor S_{ab} is defined to equal the right hand side of equation (2.1), analogous to the commonly used Einstein tensor $G_{\alpha\beta}$

$$\boldsymbol{S} = 2\boldsymbol{T} - T\boldsymbol{g}. \tag{2.46}$$

In terms of $\boldsymbol{S}$ the right hand side of the field equations are

$$S_{tt} = (\mu + 3p)e^{2U} \tag{2.47}$$

$$S_{ta} = (\mu + 3p)e^{2U} f_a \tag{2.48}$$

$$S_{ab} = (\mu + 3p)e^{2U} f_a f_b - (\mu - p)e^{-2U} \gamma_{ab}. \tag{2.49}$$

The Ricci tensor expressions in (2.37,39,41) are simplified by writing certain components in terms of others, e.g. the components of R_{ab} are written in terms of R_{tt}, R_{ta} and R_{tb}. It is therefore convenient to write the components of $\boldsymbol{S}$ in the same manner

$$S_{tt} = (\mu + 3p)e^{2U} \tag{2.50}$$

$$S_{ta} = f_a S_{tt} \tag{2.51}$$

$$S_{ab} = f_a S_{tb} + f_b S_{ta} - f_a f_b S_{tt} - (\mu - p)e^{-2U} \gamma_{ab}. \tag{2.52}$$

The Einstein field equations are now simply formed by equating $R_{\alpha\beta}$ (equations [2.37, 39, 41]) with $S_{\alpha\beta}$ (equations [2.50–52]), giving these three expressions:

$$- e^{4U}\Delta_2(U) + \tfrac{1}{4}e^{8U}\gamma^{ab}\gamma^{ij}h_{ai}h_{bj} = e^{2U}(\mu + 3p) \tag{2.53}$$

$$f_a R_{tt} + \tfrac{1}{2}e^{4U}\gamma^{ij}(h_{ia|j} + 4h_{ia}U_{,j}) = f_a S_{tt} \tag{2.54}$$

$$\begin{aligned} &f_a R_{tb} + f_b R_{ta} - f_a f_b R_{tt} + P_{ab} - 2U_{,a}U_{,b} + \gamma_{ab}\Delta_2(U) \\ &\quad - \tfrac{1}{2}e^{4U}\gamma^{ij}h_{ai}h_{bj} = f_a S_{tb} + f_b S_{ta} - f_a f_b S_{tt} - \gamma_{ab}e^{-2U}(\mu - p). \end{aligned} \tag{2.55}$$

As $R_{\alpha\beta} = S_{\alpha\beta}$, the second and third equations are trivially simplified to be

$$\gamma^{ca}(h_{ab|c} + 4h_{ab}U_{,c}) = 0 \tag{2.56}$$

$$P_{ab} - 2U_a U_b + \gamma_{ab}\Delta_2(U) - \tfrac{1}{2}e^{4U}\gamma^{cd}h_{ac}h_{bd} = -\gamma_{ab}e^{-2U}(\mu - p). \tag{2.57}$$

Equation (2.56) can be satisfied by introducing a new one-form Z defined by the following relation

$$h_{ab} = \varepsilon_{abc}\gamma^{cd}Z_d\sqrt{-\gamma}. \tag{2.58}$$

The integrability condition (2.40) determines the components of the one-form $\boldsymbol{Z}$. The ε_{abc} symbol in the definition of $\boldsymbol{Z}$ is the Levi-Cevita antisymmetric pseudo-tensor density familiar from flat space.

$$\varepsilon_{abc} = \begin{cases} +1, & \text{even permutation of } [1,2,3]; \\ -1, & \text{odd permutation of } [1,2,3]; \\ \ \ 0, & \text{otherwise.} \end{cases} \tag{2.59}$$

As ε_{abc} is not a tensor, it does not obey the standard tensor transformation properties. However, ε_{abc} can be made into a tensor by multiplying it by $\sqrt{-\gamma}$ to define the covariant tensor quantity ϵ_{abc}. (Note the subtle change in type face for the epsilons: the non-tensor "ε" verses the tensor "ϵ.")

$$\epsilon_{abc} = \sqrt{-g}\,\varepsilon_{abc}. \tag{2.60}$$

The Levi-Cevita tensor density ε^{abc} is defined by

$$\varepsilon^{abc} = \begin{cases} -1, & \text{even permutation of } [1,2,3]; \\ +1, & \text{odd permutation of } [1,2,3]; \\ \ \ 0, & \text{otherwise.} \end{cases} \tag{2.61}$$

where a sign difference between ε^{abc} and ε_{abc}, $\varepsilon^{abc} = -\varepsilon_{abc}$, has been introduced. Furthermore, ε^{abc} can also be converted into the tensor quantity ϵ^{abc} by the following rule

$$\epsilon^{abc} = \frac{1}{\sqrt{-\gamma}}\varepsilon^{abc} = -\epsilon_{abc}. \tag{2.62}$$

As ϵ_{abc} and ϵ^{abc} are tensors, they now obey the standard tensor transformation rules[44]

$$\epsilon^{abc} = \gamma^{ai}\gamma^{bj}\gamma^{ck}\epsilon_{ijk}. \tag{2.63}$$

The seemingly strange sign choice for ε^{abc} was made so that this tensor transformation rule would be valid.

The covariant derivatives of $\boldsymbol{\epsilon}$ can be deduced by examining the covariant derivative of a single component[45]

$$\begin{aligned}\epsilon_{123|a} &= \frac{\partial}{\partial x^a}\sqrt{-\gamma}\varepsilon_{abc} - \Sigma^b{}_{1a}\epsilon_{b23} - \Sigma^b{}_{2a}\epsilon_{1b3} - \Sigma^b{}_{3a}\epsilon_{12b} \\ &= \left[(\sqrt{-\gamma}\,)_{,a} - \Sigma^b{}_{ba}\sqrt{-\gamma}\,\right]\varepsilon_{123}.\end{aligned} \tag{2.64}$$

In a coordinate basis, such as the coordinate system used in this paper,

$$\Sigma^b{}_{ba} = \left[\ln(\sqrt{-\gamma}\,)\right]_{,a} = \frac{(\sqrt{-\gamma}\,)_{,a}}{\sqrt{-\gamma}}. \tag{2.65}$$

Inserting this result into (2.64) gives the identity (valid in a coordinate basis) that all covariant derivatives of $\boldsymbol{\epsilon}$ are zero

$$\epsilon_{abc|d} = 0. \tag{2.66}$$

The $\boldsymbol{Z}$ one-form is now defined by this expression for the integrability condition

$$\gamma^{ab}Z_{a|b} = 0. \tag{2.67}$$

The three field equations now must also be expressed in terms of Z. The first field equation, equation (2.53), becomes

$$\Delta_2(U) + \tfrac{1}{2}e^{4U}\gamma^{ab}Z_aZ_b = -e^{-2U}(\mu + 3p). \tag{2.68}$$

[44] Carmeli 1982 pp. 37–38.
[45] Misner *et al.* 1973 p. 501.

The second field equation (2.56) contains the term $h_{ab|c}$,

$$h_{ab|c} = \epsilon_{abi}\gamma^{ij}Z_{j|c} = \epsilon_{abi}\gamma^{ij}(Z_{j,c} - \Sigma^k{}_{jc}Z_k). \tag{2.69}$$

Inserting (2.69) into (2.56) gives the second field equation in terms of Z

$$\epsilon^{acj}(Z_{j,c} + 4Z_jU_{,c}) = 0. \tag{2.70}$$

Equation (2.57) contains the term $\gamma^{cd}h_{ac}h_{bd}$ which becomes

$$\gamma^{cd}h_{ac}h_{bd} = \gamma^{cd}\gamma^{ei}\gamma^{jk}\epsilon_{ace}\epsilon_{bdj}Z_iZ_k. \tag{2.71}$$

The right hand side of equation (2.71) can be simplified by multiplying through with $\delta^a{}_m = \gamma^{an}\gamma_{nm}$, allowing the term to be expressed as

$$\gamma^{cd}h_{mc}h_{bd} = \gamma_{nm}\gamma^{jk}\epsilon^{ind}\epsilon_{jbd}Z_iZ_k, \tag{2.72}$$

which can be expanded and simplified with the familiar identity[46]

$$\epsilon^{ind}\epsilon_{jbd} = -\delta^i{}_j\delta^n{}_b + \delta^i{}_b\delta^n{}_j. \tag{2.73}$$

By combining several of the above equations, the third field equation (2.57) becomes

$$\begin{aligned} P_{ab} - 2U_{,a}U_{,b} + \gamma_{ab}\Delta_2(U) + \tfrac{1}{2}e^{4U}\gamma_{ab}\gamma^{ij}Z_iZ_j - \tfrac{1}{2}e^{4U}Z_aZ_b \\ = -(\mu - p)e^{-2U}\gamma_{ab}. \end{aligned} \tag{2.74}$$

By substituting the first field equation into the third, the third field equation (2.74) simplifies to

$$P_{ab} - 2U_{,a}U_{,b} - \tfrac{1}{2}e^{4U}Z_aZ_b = 4pe^{-2U}\gamma_{ab}. \tag{2.75}$$

The integrability condition, equation (2.67), and field equations, equations (2.68,70,75) now are

$$\gamma^{ab}Z_{a|b} = 0 \tag{2.76}$$
$$\Delta_2(U) + \tfrac{1}{2}e^{4U}\gamma^{ab}Z_aZ_b = -(\mu + 3p)e^{-2U} \tag{2.77}$$
$$\epsilon^{acj}(Z_{j,c} + 4Z_jU_{,c}) = 0 \tag{2.78}$$
$$P_{ab} - 2U_{,a}U_{,b} - \tfrac{1}{2}e^{4U}Z_aZ_b = 4pe^{-2U}\gamma_{ab}. \tag{2.79}$$

[46] The sign of this identity is reversed from its common form due to the definition of the contravariant Levi-Cevita tensor density in equation (2.61).

Equation (2.78) can be satisfied by defining a "twist" potential ψ such that

$$Z_a = e^{-4U}\psi_{,a}. \tag{2.80}$$

After making this change of variables, the field equations for the general metric finally become

$$\Delta_2(\psi) - 4\Delta_1(\psi, U) = 0 \tag{2.81}$$
$$\Delta_2(U) + \tfrac{1}{2}e^{-4U}\Delta_1(\psi) = -(\mu + 3p)e^{-2U} \tag{2.82}$$
$$P_{ab} = 2U_{,a}U_{,b} + \tfrac{1}{2}e^{-4U}\psi_{,a}\psi_{,b} + 4pe^{-2U}\gamma_{ab}. \tag{2.83}$$

Equation (2.78) reduces to $0 = 0$ when changing variables, and has been dropped from the list of field equations.

Equations (2.81–83) are the field equations for Harrison's general metric given in (2.23). These equations now need to be written for the metric (2.20), the metric used in this paper. The metric (2.20) has two isometries, requiring that all metric functions be independent of t and ϕ.

$$ds^2 = -f(dt - Qd\phi)^2 + \frac{1}{f}\left[s^2 d\phi^2 + e^{2\zeta}(d\rho^2 + dz^2)\right]. \tag{2.84}$$

Comparing (2.20) with (2.23) identifies the gamma components and their inverses

$$\begin{array}{ll} \gamma_{\phi\phi} = -s^2 & \gamma^{\phi\phi} = -\dfrac{1}{s^2} \\ \gamma_{\rho\rho} = -e^{2\zeta} & \gamma^{\rho\rho} = -e^{-2\zeta} \\ \gamma_{zz} = -e^{2\zeta} & \gamma^{zz} = -e^{-2\zeta}, \end{array} \tag{2.85}$$

and the metric components are

$$\begin{array}{ll} g_{tt} = -f & g^{tt} = \dfrac{fQ^2}{s^2} - \dfrac{1}{f} \\ g_{t\phi} = -fQ & g^{t\phi} = -\dfrac{fQ}{s^2} \\ g_{\rho\rho} = g_{zz} = \dfrac{1}{f}e^{2\zeta} & g^{\rho\rho} = g^{zz} = fe^{-2\zeta}. \end{array} \tag{2.86}$$

The γ metric is diagonal, and Landau and Lifshitz developed simple expressions for the Christoffel symbols and Ricci tensor components for diagonal metrics, which were used to calculate $\Sigma^i{}_{jk}$ and P_{ij}. These expressions are listed in Appendix C. The non-zero Christoffel symbols associated with γ_{ij} are

$$\Sigma^{\phi}{}_{\phi A} = \frac{1}{s} s_{,A} \tag{2.87}$$

$$\Sigma^{A}{}_{\phi\phi} = -s e^{-2\zeta} s_{,A} \tag{2.88}$$

$$\Sigma^{A}{}_{BB} = -\zeta_{,A} \qquad (A \neq B) \tag{2.89}$$

$$\Sigma^{B}{}_{BA} = \zeta_{,A}. \tag{2.90}$$

For future reference the complete non-zero Christoffel symbols are

$$\Gamma^{t}{}_{tA} = \frac{1}{2}\left(\frac{f^2 Q}{s^2} Q_{,A} + \frac{f_{,A}}{f}\right) \tag{2.91}$$

$$\Gamma^{t}{}_{\phi A} = \frac{1}{2}\left(Q_{,A} + \frac{2Q}{f} f_{,A} + \frac{f^2 Q^2}{s^2} Q_{,A} - \frac{2Q}{s} s_{,A}\right) \tag{2.92}$$

$$\Gamma^{\phi}{}_{tA} = -\frac{1}{2}\frac{f^2}{s^2} Q_{,A} \tag{2.93}$$

$$\Gamma^{\phi}{}_{\phi A} = -\frac{1}{2}\left(\frac{f^2 Q}{s^2} Q_{,A} + \frac{f_{,A}}{f} - \frac{2}{s} s_{,A}\right) \tag{2.94}$$

$$\Gamma^{A}{}_{tt} = \frac{1}{2} f e^{-2\zeta} f_{,A} \tag{2.95}$$

$$\Gamma^{A}{}_{t\phi} = \frac{1}{2} f e^{-2\zeta} (f_{,A} Q + f Q_{,A}) \tag{2.96}$$

$$\Gamma^{A}{}_{\phi\phi} = \frac{1}{2} f e^{-2\zeta}\left(f_{,A} Q^2 + 2 f Q Q_{,A} + \frac{f^2}{s^2} f_{,A} - \frac{2s}{f} s_{,A}\right) \tag{2.97}$$

$$\Gamma^{A}{}_{BB} = -\frac{1}{2}\left(2\zeta_{,A} - \frac{f_{,A}}{f}\right) \qquad (A \neq B) \tag{2.98}$$

$$\Gamma^{B}{}_{BA} = \frac{1}{2}\left(2\zeta_{,A} - \frac{f_{,A}}{f}\right). \tag{2.99}$$

The differential parameters from equations (2.26–28) are

$$\Delta_1(F, G) = e^{-2\zeta}(F_{,\rho} G_{,\rho} + F_{,z} G_{,z}) \tag{2.100}$$

$$\Delta_2(F) = e^{-2\zeta}\left(F_{,\rho\rho} + F_{,zz} + \tfrac{1}{s} s_{,\rho} F_{,\rho} + \tfrac{1}{s} s_{,z} F_{,z}\right). \tag{2.101}$$

And the γ-Ricci tensor components are

$$P_{\phi\phi} = -se^{-2\zeta}(s_{,\rho\rho} + s_{,zz}) \tag{2.102}$$
$$P_{\rho\rho} = \frac{1}{s}(\zeta_{,\rho}s_{,\rho} - \zeta_{,z}s_{,z} - s_{,\rho\rho}) - \zeta_{,\rho\rho} - \zeta_{,zz} \tag{2.103}$$
$$P_{zz} = \frac{1}{s}(\zeta_{,z}s_{,z} - \zeta_{,\rho}s_{,\rho} - s_{,zz}) - \zeta_{,\rho\rho} - \zeta_{,zz} \tag{2.104}$$
$$P_{\rho z} = \frac{1}{s}(s_{,z}\zeta_{,\rho} + s_{,\rho}\zeta_{,z} - s_{,\rho z}) \tag{2.105}$$
$$P_{\phi\rho} = P_{\phi z} = 0. \tag{2.106}$$

Combining all this information one finally arrives at the full field equations:

$$\psi_{,\rho\rho} + \psi_{,zz} + \frac{1}{s}(s_{,\rho}\psi_{,\rho} + s_{,z}\psi_{,z}) - \frac{2}{f}(\psi_{,\rho}f_{,\rho} + \psi_{,z}f_{,z}) = 0 \tag{2.107}$$
$$f_{,\rho\rho} + f_{,zz} + \frac{1}{f}(\psi_{,\rho}^2 + \psi_{,z}^2 - f_{,\rho}^2 - f_{,z}^2) + \frac{1}{s}(s_{,\rho}f_{,\rho} + s_{,z}f_{,z}) = 2e^{2\zeta}(\mu + 3p) \tag{2.108}$$
$$\frac{1}{s}(s_{,z}\zeta_{,\rho} + s_{,\rho}\zeta_{,z} - s_{,\rho z}) = \frac{1}{2f^2}(f_{,\rho}f_{,z} + \psi_{,\rho}\psi_{,z}) \tag{2.109}$$
$$\frac{1}{s}(s_{,\rho\rho} + s_{,zz}) = \frac{4}{f}pe^{2\zeta} \tag{2.110}$$
$$\frac{1}{s}(\zeta_{,\rho}s_{,\rho} - \zeta_{,z}s_{,z} - s_{,\rho\rho}) - \zeta_{,\rho\rho} - \zeta_{,zz} = \frac{1}{2f^2}(f_{,\rho}^2 + \psi_{,\rho}^2) - \frac{4}{f}pe^{2\zeta} \tag{2.111}$$
$$\frac{1}{s}(\zeta_{,z}s_{,z} - \zeta_{,\rho}s_{,\rho} - s_{,zz}) - \zeta_{,\rho\rho} - \zeta_{,zz} = \frac{1}{2f^2}(f_{,z}^2 + \psi_{,z}^2) - \frac{4}{f}pe^{2\zeta}. \tag{2.112}$$

The metric function Q has been replaced by the twist potential ψ in writing the field equations. While the field equations are complete, physical constraints are commonly formulated in terms of the metric components, and it is useful to express Q explicitly in terms of ψ. From equation (2.78) one calculates that

$$Q_{,\rho} = \frac{s}{f^2}\psi_{,z} \tag{2.113}$$
$$Q_{,z} = -\frac{s}{f^2}\psi_{,\rho}. \tag{2.114}$$

The Bianchi identities $R^{\alpha}{}_{\beta[\gamma\delta;\mu]} = 0$, and the stress-energy tensor identity $T^{\alpha\beta}{}_{;\beta} = 0$, are further conditions that the metric functions must obey.

Their conditions are not always unique, and in this case they both yield the same result (although the Bianchi identities are much more difficult to calculate) that

$$\frac{dp}{df} = -\frac{1}{2f}(\mu + p). \tag{2.115}$$

The Bianchi identities, combined with the equation of state $p = p(\mu)$, show that p and μ can be expressed as functions only of f, which allows choosing f as a coordinate[47] in expressions for μ and p

$$p = p(f) \tag{2.116}$$

$$\mu = \mu(f). \tag{2.117}$$

This concludes the derivation of the field equations. Their final form is given in equations (2.107–112). The Bianchi identities give the important result in (2.116–117) that $\mu = \mu(f)$ and $p = p(f)$. Having derived the field equations for the rigidly rotating, axially symmetric, stationary perfect fluid, the next step is to find the Lie-Bäcklund symmetries, the topic of the next chapter.

[47] Harrison B K 1970 *Phys. Rev. D* **1** 2269.