ONE

Introduction

The study of rotating, axially symmetric spacetimes in general relativity has attracted considerable attention due to their significant astrophysical importance—examples range from planets to stars to black holes. The Kerr family of solutions in general relativity describes the gravitational field or spacetime geometry outside of such an object,[1] but much less is known about the spacetime geometry of the body's interior, the region of spacetime containing the matter distribution. In fact, no solutions for the interior geometry of a rotating, axially symmetric body have been found that satisfy all physical boundary conditions.[2] The reason is simple: general relativity's field equations are extremely difficult to solve. The currently known solutions were found by simplifying the equations with *ad hoc* assumptions—not to mention a great deal of work and skill.

This paper seeks to solve the Einstein field equations for the interior of an axially symmetric, stationary, rotating, relativistic body. Stationary rotation means that the rotation is constant in time. A relativistic body means that the gravitational field's nonlinear characteristics, self-gravity effects, as well as the internal pressure's contribution to the field are important. These characteristics are in contrast to standard Newtonian gravitation, which is linear and a function only of the source's rest mass. Relativistic considerations

[1] Technically, the Kerr family of solutions describes the exterior field for sources with particular multipole moments. The exterior field of other bodies is very similar. See Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Space-Time* (Cambridge: Cambridge University Press) p. 161.

[2] This sentence assumes a perfect fluid matter distribution. A few perfect fluid interior solutions are known, but no single solution satisfies all criteria for a physical interpretation. Some of these solutions and the physical criteria are listed in § 1.2.

rule out planets as a physical picture, because their relativistic effects are too small. An extremely dense and massive star, such as a neutron star, provides the physical motivation for the equations and boundary conditions, as well as a good mental picture of the rotating body to guide the reader in the mathematical thicket that follows. This introduction first discusses the inherent difficulty in solving the field equations for the general case, and the employment of Lie-Bäcklund symmetries to simplify the equations. The introduction concludes by reviewing literature specific to stationary, axisymmetric, fluid spacetimes.

1.1 The Interior Field Equations

The Einstein field equations are elegantly simple (and simply elegant) when expressed in covariant form

$$\boldsymbol{G} = \frac{8\pi G}{c^4}\boldsymbol{T}, \tag{1.1}$$

but solving the equations requires one to write them in a specific coordinate system—virtually a fall from grace. This fall's consequences are severe: the elegance of covariance is marred as the field equations become ten complex, nonlinear, coupled, partial differential equations. In vacuum these ten equations contain the ten independent components of the metric tensor; the interior case is further complicated with the addition of unknown functions describing the matter distribution's mass density, four-velocity and internal stresses.

Perfect fluids provide the most realistic, yet simple—and by far the most common—models for real matter in general relativity. Thus little is lost in this discussion by specializing immediately to the perfect fluid case. A perfect fluid is characterized by its mass density, four-velocity and an isotropic pressure.[3] Thus the general perfect fluid field equations are partial differential equations containing sixteen arbitrary functions of four independent variables: the ten metric coefficients, the perfect fluid's mass density, the four four-velocity components and isotropic pressure. In facing

[3] The perfect fluid's *properties* are detailed in § 2.1. The mass density and isotropic pressure are measured in the fluid's rest frame.

such desperate situations, use of physical symmetries is a proven and powerful tool. The vacuum field equations have succumbed to symmetry analyses to yield useful solutions; the interior case has proven more difficult, and no physically realistic solution has been found.

Interior studies continue on the dual fronts of numeric and analytic work. Numeric results provide an important foothold in comprehending relativistic, axially symmetric, rotating bodies. Numeric algorithms also allow one to employ more complicated and realistic matter models, and these techniques have been very successful in modeling stellar interiors.[4] While numeric work is important, analytic results are not only more aesthetically pleasing, but may be more versatile in exploring extreme relativistic cases and could perhaps be applicable to a wider variety of matter geometries.

As mentioned above, the use of physical symmetries alone has not been sufficient to find an exact solution for a rotating, axially symmetric, relativistic body. Progress on the analytic problem has been made chiefly by incorporating *ad hoc* assumptions, or *Ansätze,* chosen to simplify the equations or provide additional constraints, allowing a solution to be found. *Ansätze* appear in a variety of styles and types, though common *Ansätze* can be ordered in two classes:

(I) Imposing conditions
  (a) by requiring the spacetime to be algebraically special,
  (b) by making assumptions about the system's physical characteristics;

(II) Directly simplifying the differential equations.

The first class of *Ansätze* simplifies the field equations by supplying additional information in the form of condition equations. These extra conditions on the metric components often enable at least a partial solution. An algebraically special solution refers to Petrov's classification of spacetimes by their null directions.[5] Interestingly, the physically most important family of

[4] Friedman J L, Ipser J R and Parker L 1986 *Ap. J.* **304** 115. Bonazzola S and Schneider J 1974 *Ap. J.* **191** 273. Butterworth E M and Ipser J R 1976 *Ap. J.* **204** 200.

[5] Stephani H 1990 *General Relativity: An Introduction to the Theory of the Gravitational Field* (New York: Cambridge University Press) pp. 171–194. Wald R M 1984 *General Relativity* (Chicago: University of Chicago Press) pp. 179–180.

exterior solutions, the Kerr family,[6] is Petrov type D.[7] Thus, the assumption that the interior spacetime is also type D is frequently made. Physical characteristics offer constraints that might include an irrotational perfect fluid, or rigid rotation.[8]

The second class of *Ansätze* seeks to simplify the extant equations. These simplifications may include modifying the equations in an *ad hoc* fashion such that a solution can be found, for example arbitrarily setting constants to zero. The second class of *Ansätze* also includes any assumptions regarding the equations' solutions. For example one could assume a specific form for functions appearing in the differential equations. The separation of variables technique used in solving linear partial differential equations is a common example of this kind of *Ansatz.* Another technique is to look for similarity solutions. These solutions result from special transformations designed to simplify the differential equations. This paper uses this technique, and similarity transformations are discussed below and in Chapter Three. The choice of *Ansätze* is wide and left to the whim of individual researchers, who may employ multiple *Ansätze* at the same time. Of course the danger lies in oversimplifying the equations, producing only trivial solutions.

This paper investigates the interior of a stationary, axially symmetric, rigidly rotating, relativistic perfect fluid. Two *Ansätze* simplify the field equations, one from each of the classes above, namely:

(i) rigid rotation (Class I);

(ii) a special functional dependence for the metric functions (Class II).

The rigid rotation assumption allows the introduction of comoving coordinates to simplify expressions involving the perfect fluid. A specific functional dependence for the metric components means that the independent variables appear in the metric functions only in special combinations. As a concrete example, one might assume that the independent variables (ρ, z) appear in the metric functions only in the specific combination ρ/z. The functional

[6] The Kerr family contains the Schwarzschild, Reissner-Nordström, Kerr and the Kerr-Newman solutions.

[7] Wald R M 1984 p. 313.

[8] These *Ansätze* are not to be confused with physical constraints, which are conditions a solution must meet to be physically relevant. Examples of these constraints are found in § 1.2.

dependence will be chosen to correspond to similarity variables, and the resulting solution—if indeed one can be found—will be a special case of the general solution, called a similarity solution.[9] The use of similarity techniques is outlined below.

Similarity transformations are a class of transformations that can be used to simplify differential equations, and often represent the equations' invariance properties, such as invariance under scale changes. Similarity transformations are transformations of the independent and dependent variables and their derivatives designed to eliminate at least one independent variable from the equations. This process is commonly called "order reduction" in the literature. The new variables are called similarity variables, and although the number of independent variables in the equations is reduced, the number of independent variables required to describe the system remains the same. Naturally, one difficulty in using similarity transformations is finding a proper set of variables that enables a reduction of variables in the equations. As discussed in the next paragraph, symmetries and similarity transformations are closely tied together. These symmetries offer a systematic approach to selecting the proper set of similarity variables.

Similarity transformations do not exist for arbitrary differential equations, but are only possible when the equations possess underlying symmetries, or invariance properties. That similarity transformations depend on symmetries is not surprising, since in general symmetries imply a reduction of the independent variables in a problem from the general case. For example, an arbitrary function F in cylindrical coordinates is generally a function of all three coordinates, $F = F(\rho, \phi, z)$. Assuming that F is axially symmetric eliminates the ϕ dependence in F, reducing by one the number of independent variables, $F = F(\rho, z)$. While the cylindrical symmetry used in this example can be considered a physical symmetry, the mathematical symmetries behind similarity transformations are the more abstract Lie-Bäcklund symmetries. Lie-Bäcklund symmetries are transformations in the space of independent and dependent variables and their derivatives. The Lie-Bäcklund symmetries of differential equations are those transformations that leave the equations'

[9] Stephani H 1992 *Differential Equations: Their Solution Using Symmetries* (Cambridge: Cambridge University Press) pp. 170–183, 248–250.

form invariant. These symmetry generators, by definition, map solutions of the differential equations into other solutions, while the equations themselves are invariant under the symmetry generator's operation. Stephani notes that the Einstein equations' Lie-Bäcklund symmetries are the "hidden" symmetries commonly referred to in the literature.[10]

Differential equations are easier to solve when similarity techniques can be used, but the method also extracts a price: the similarity solution is not the general solution. Whether or not the similarity solution is physically important depends on whether the solution satisfies the physically motivated boundary conditions. If the similarity variables reflect some physical symmetry in the system, then the solutions are often compatible with the boundary conditions. But there is no guarantee that similarity variables that are compatible with the boundary conditions produce solutions that also are compatible with the boundary conditions. Thus one can not know from the outset whether introducing similarity variables will yield a physically interesting solution. Only after the solution is known can the boundary conditions be tested. A solution that meets the required boundary conditions justifies the *Ansatz* of choosing the special functional dependence for the metric functions.

Two physical symmetries, stationarity and axial symmetry, simplify the equations while maintaining a realistic astrophysical model. The two physical symmetries reduce the independent coordinates (t, ϕ, ρ, z) by two, leaving (ρ, z) as the independent coordinates. Lie-Bäcklund symmetries will then be used to choose similarity variables, allowing one independent variable to be eliminated from the field equations, accomplishing the reduction from partial differential equations to ordinary differential equations. The simplification is that the field equations are now ordinary differential equations, as the methods for solving ordinary differential equations are both more abundant and simpler. The equations' actual form is, unfortunately, much more complicated. These equations could not be solved except in one extremely simple, and physically unrealistic case. The Lie-Bäcklund symmetry generator will

[10] Stephani H 1988 *J Math. Phys.* **29** 1650.

be found using Harrison and Estabrook's modern approach, using differential forms,[11] the calculations facilitated with the computer algebra program *Maple V*.

1.2 Literature Review

As previously mentioned, employing *ad hoc* assumptions has enabled progress in extracting exact solutions from the field equations. An *Ansatz* that yields a physical solution must be based on the system's physical or geometric characteristics. Thus *Ansätze* must be chosen with care. Only a handful of interior solutions for rotating, axially symmetric perfect fluids with no other symmetries are known. Unfortunately, none of the known solutions have an astrophysical interpretation, perhaps because currently the employed *Ansätze* are to some degree mathematical simplifications, unrelated to the system's inherent physical properties. Indeed, *Ansätze* of Class II are often mathematical simplifications. Herlt lists three properties of physically relevant solutions:

(i) The solution has a bounding surface where the pressure vanishes. The pressure and mass density are positive inside this boundary.

(ii) The solution has a regular rotation axis. The orbits are compact with a periodicity of 2π.

(iii) The solution can be matched to an axisymmetric, asymptotically flat vacuum solution.[12]

Several known solutions do not meet the first criterion, and no known interior, axisymmetric solution has been matched to an asymptotically flat exterior solution. These solutions are valuable, however, as they develop new techniques in the continuing search for physically important solutions.

Several authors have investigated rotating perfect fluid systems. The second *Ansatz* used in this paper—using a Lie-Bäcklund symmetry to propose an assumed functional dependence—has not appeared in the literature. The following is a representative list of known interior solutions, with occasional comments. The first group contains rigidly rotating perfect fluid solutions; the differential rotation solutions are treated at the end.

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[11] Harrison B K and Estabrook F B 1971 *J Math. Phys.* **12** 653.

[12] Herlt E 1988 *Gen. Rel. Grav.* **20** 635.

Wahlquist in 1968 found the first interior solution for a rigidly rotating, axisymmetric, stationary perfect fluid.[13] Wahlquist's solution is Petrov type D, and has two unexpected properties, namely that the fluid has a finite boundary in the shape of a *prolate* spheroid,[14] and an equation of state $\mu + 3p =$ constant. In a recently published article detailing this solution's derivation, Wahlquist discusses these features and their possible physical interpretations.[15] Regarding the solution's prolate boundary, Wahlquist notes that electromagnetic fields, which most likely occur in rotating systems, might allow a larger selection of admissible shapes. The equation of state at first looks unphysical because the pressure must decrease as the mass density increases. However, Wahlquist argues that the matter's intrinsic thermodynamic properties cannot be inferred from the average, mechanical quantities μ and p. Furthermore, Wahlquist argues that a diagonal stress-energy tensor does not necessarily imply a perfect fluid matter distribution. Other matter distributions, under intense gravitational fields for long periods, may develop an isotropic stress equilibrium. This allows the solution and equation of state to be applied to non-perfect-fluid matter distributions. This could be advantageous because matter with solid regions may have negative pressures.

In 1984 Kramer found a solution with characteristics similar to Wahlquist's solution.[16] Senovilla showed that Kramer's solution could be derived from Wahlquist's in a limiting procedure,[17] and in a following paper found all metrics of type D with equation of state $\mu + 3p =$ const.[18] Herlt found a solution for a fluid of vanishing vorticity, or equivalently, by requiring that the real and imaginary parts of the Ernst potential be functionally dependent.[19]

[13] Wahlquist H D 1968 *Phys. Rev.* **172** 1291.

[14] Numeric stellar solutions have *oblate* spheroidal boundaries, as one may expect from Newtonian analogy. See, for example, the references listed in footnote four.

[15] Wahlquist H D 1992 *J Math. Phys.* **33** 304.

[16] Kramer D 1984 *Class. Quantum Grav.* **1** L3.

[17] Senovilla J M M 1987 *Class. Quantum Grav.* **4** L115.

[18] Senovilla J M M 1987 *Phys. Lett.* **123A** 211.

[19] Herlt E 1988 *Gen. Rel. Grav.* **20** 635.

In an important paper Bonanos and Sklavenites use a metric with a $g_{\rho z}$ component, wherein Einstein's equations reduce to two partial differential equations for the metric coefficients.[20] Kyriakopoulos, Sklavenites and Bonanos have used these coordinates, combined with different *Ansätze,* to find several solutions. Sklavenites, requiring the magnetic Weyl tensor to vanish, found a solution with equation of state $\mu = p + \text{constant}$.[21] Kyriakopoulos found a class of solutions with equation of state $p = \gamma\mu$, where the constant γ is greater than zero.[22] Kyriakopoulos found another class of solutions[23] and investigated matching them to exterior solutions.[24] Sklavenites found a class of solutions with equation of state $\mu = (4\nu - 3)p$, where $\nu = \text{constant}$.[25] Additional solutions with the equation of state $\mu + 3p = 0$ have been found by Kyriakopoulos[26] and Bonanos and Kyriakopoulos.[27]

While much more work has been done with rigid rotation, the differential rotation case has not been neglected. Stephani and Grosso investigated fluids with the four velocity spanned by two commuting Killing vectors, but did not find a solution.[28] Senovilla found a family of solutions with the equation of state $\mu = p + \text{constant}$.[29] Finally, Chinea and González-Romero found a solution with irrotational fluid motion[30] and a second solution up to a final ordinary differential equation for a differentially rotating perfect fluid with vanishing vorticity.[31]

[20] Bonanos S and Sklavenites D 1985 *J Math. Phys.* **26** 2275.
[21] Sklavenites D 1985 *J Math. Phys.* **26** 2279.
[22] Kyriakopoulos E 1987 *J Math. Phys.* **28** 2162.
[23] Kyriakopoulos E 1988 *Gen. Rel. Grav.* **20** 427.
[24] Kyriakopoulos E 1989 *Gen. Rel. Grav.* **21** 125.
[25] Sklavenites D 1992 *Gen. Rel. Grav.* **24** 935.
[26] Kyriakopoulos E 1992 *Class. Quantum Grav.* **9** 217.
[27] Bonanos S and Kyriakopoulos E 1994 *Class. Quantum Grav.* **11** L23.
[28] Stephani H and Grosso R 1989 *Class. Quantum Grav.* **6** 1673.
[29] Senovilla J M M 1992 *Class. Quantum Grav.* **9** L167.
[30] Chinea F J and González-Romero L M 1990 *Class. Quantum Grav.* **7** L99.
[31] Chinea F J and González-Romero L M 1992 *Class. Quantum Grav.* **9** 1271.