A Search for an Interior Solution in General Relativity Using Lie-Bäcklund Symmetries

A Thesis

Presented to the

Department of Physics and Astronomy

Brigham Young University

In Partial Fulfillment

of the Requirements for the Degree

Master of Science

by

David Wayne Neilsen

August 1995

This thesis by David Wayne Neilsen is accepted in its present form by the Department of Physics and Astronomy of Brigham Young University as satisfying the thesis requirement for the degree of Master of Science.

B. Kent Harrison, Committee Chair

S. Neil Rasband, Committee Member

Date

Dorian M. Hatch, Department Chair

Then let the relativist rejoice in the ivory tower where he has peace to seek understanding of Einstein's theory as long as the busy world is satisfied to do its jobs without him. Let him be satisfied with the difficult task of gaining for himself as much understanding as he can, and the still more difficult task of transmitting to others the scraps of understanding he has been able to gain.

—J. L. Synge, *Relativity: The General Theory*

Contents

Appendices

Acknowledgments

This thesis would not have an appropriate beginning without thanking my advisor, Professor B. Kent Harrison, who spent many hours patiently teaching me general relativity.

I also express my love and gratitude to my wife, Tracianne, for her unfailing support during this project.

Conventions

This paper contains three different types of equations: tensor equations, exterior differential form equations, and differential equations. Choosing a consistent notation convention requires balancing the needs of each equation type, while striving for uncluttered, simple expressions. In one area, index convention, a satisfactory compromise between convention needs could not be obtained. Thus indices follow one convention for tensor equations, and another for differential form equations. This is only possible because these two types of equations maintain a "spacelike" separation: tensor equations are essentially confined to Chapter 2, while differential form equations are similarly confined to Chapter 3. The index convention for tensor equations is outlined below; index use with differential forms is more specialized and is explained as the indices are introduced. Specifically, this paper's conventions are:

(i) Units are chosen such that $c = 1$ and $G = 1/4\pi$.

(ii) The metric signature is $(- + + +)$.

(iii) All repeated indices sum, and in tensor equations the indices follow these conventions:

(a) Greek indices (α, β, γ, etc.) sum over the four dimensions of spacetime $0, 1, 2, 3$;

(b) Lower-case Latin indices (a, b, c, etc.) sum over the three spacelike dimensions $1, 2, 3$;

(c) Upper-case Latin indices (A, B, C, etc.) sum over $2, 3$.

(iv) "Canonical" cylindrical coordinates (t, ϕ, ρ, z) label the four spacetime dimensions, and do not sum when used as indices.

(v) Commas in indices denote ordinary differentiation, e.g.

$$T^{\alpha\beta}{}_{,\gamma} = \frac{\partial}{\partial x^\gamma} T^{\alpha\beta}. \tag{0.1}$$

(vi) Semicolons in indices denote covariant differentiation, e.g.

$$V^\alpha{}_{;\gamma} = V^\alpha{}_{,\gamma} + \Gamma^\alpha{}_{\epsilon\gamma} V^\epsilon. \tag{0.2}$$

(vii) A vertical stroke '$|$' in indices denotes covariant differentiation with respect to a restricted, spatial metric $\boldsymbol{\gamma}$.

(viii) Square brackets in indices denote the anti-symmetrization operation

$$T^{[\alpha\beta]} = \frac{1}{2!}(T^{\alpha\beta} - T^{\beta\alpha}) \qquad (0.3).$$

(ix) Boldface italic letters represent tensor quantities, e.g. the Ricci tensor $\boldsymbol{R}$, an arbitrary vector $\boldsymbol{\chi}$, etc.

(x) The Christoffel symbols are defined by

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}). \tag{0.4}$$

(xi) The Riemann tensor is defined by

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}. \tag{0.5}$$

(xii) The Ricci tensor $\boldsymbol{R}$ is defined by

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}. \tag{0.6}$$

(xiii) The Einstein field equations are

$$R_{\mu\nu} = 2\,T_{\mu\nu} - T g_{\mu\nu}. \tag{0.7}$$

(xiv) The wedge product "$\wedge$" in differential form equations is often suppressed. Thus a wedge product is understood between repeated forms, e.g.

$$d\alpha\, d\beta \equiv d\alpha \wedge d\beta. \tag{0.8}$$

A Search for an Interior Solution in General Relativity
Using Lie-Bäcklund Symmetries

David Wayne Neilsen

Department of Physics and Astronomy

M.S. Degree, August 1995

ABSTRACT

The Einstein field equations describing a stationary, axially symmetric, rigidly rotating, relativistic, perfect fluid are reduced with a similarity transformation to ordinary differential equations. A Lie-Bäcklund symmetry analysis is used to obtain the similarity variables. Approaches to a solution of the equations are discussed. A simple case allows the equations to be solved, yielding a non-rotating spacetime with cartesian symmetry.

COMMITTEE APPROVAL: ______________________________
B. Kent Harrison, Committee Chair

____
S. Neil Rasband, Committee Member

____
Dorian M. Hatch, Department Chair