Appendix B

Problems with Liesymm

Computer algebra programs offer tremendous capabilities to the modern researcher, saving literally hours—and pages—of tedious algebra. However, the software's limitations must be thoroughly understood to avoid costly and hard-to-find errors.

The `liesymm` package was tested extensively with simple nonlinear differential equations before being employed to solve for the field equations' Lie-Bäcklund symmetry generators. To illustrate difficulties with the `liesymm's` automatic routines `determine` and `autosimp` the Korteweg-de Vries (KdV) equation will suffice. It must be noted that these results were obtained with Release 2, the most current version available at the time. Apparently some bugs were fixed with Release 3, as these programs do not produce the wrong answers when run on Release 3. However, the release notes for Release 3 do not document any changes in the `liesymm` package.

The Kortweg-de Vries equation

$$u_{,t} + u u_{,x} + \epsilon u_{,xxx} = 0 \tag{B.1}$$

was analyzed three times by `liesymm's` automatic routines using different values for ϵ, with the results compared to the symmetry generators given by Harrison and Estabrook.[1] The `liesymm` package did not obtain Harrison and Estabrook's result on any of the three runs. Additionally, the invariance properties should be independent of ϵ, however, `liesymm` produced two different invariance groups for two different choices of ϵ.

[1] Harrison B K and Estabrook F B 1971 *J Math. Phys.* **12** 653.

These programs were run on Maple V Release 2 licensed to Brigham Young University on an Hewlett-Packard 9000/735 workstation. In each case the program was of the form:

```
with(liesymm):
eq1 := Diff(u(t,x),t) + u(t,x)*Diff(u(t,x),x) +
     k*Diff(u(t,x),x$3) = 0;
eq2 := determine(eq1,V,u(t,x),w):
eq3 := autosimp(eq2);
quit;
```

In Run A ϵ is set equal to the arbitrary constant k. In Run B, $\epsilon = 1$, and in Run C, $\epsilon = 12$. The Results of Cases A and B are:

$$V^t = C_1 + 3C_3 t + \tfrac{3}{2} C_6 t^2 \tag{B.2}$$

$$V^x = C_2 + C_3 x + C_4 t + C_6 x t \tag{B.3}$$

$$V^u = C_4 + C_5 u + C_6 (x - ut). \tag{B.4}$$

The results from Case C are

$$V^t = C_1 + 3C_3 t + \tfrac{3}{2} C_6 t^2 \tag{B.5}$$

$$V^x = C_2 + C_3 x + C_4 t + C_6 x t \tag{B.6}$$

$$V^u = C_4 - 2C_3 u + C_6 (x - 2ut). \tag{B.7}$$

Finally, Harrison and Estabrook report the invariance group to be

$$V^t = C_1 + 3C_3 t \tag{B.8}$$

$$V^x = C_2 + C_3 x + C_4 t \tag{B.9}$$

$$V^u = C_4 - 2C_3 u. \tag{B.10}$$

Comparing the above sets of equations shows that neither set generated by Maple is correct.

In a short explanation of the `liesymm` package, Heck notes that in Case C *Maple* returns the following list[2]

```
case_c := {} &where {

                                  2
     V1(t, x, u) = C9 - 3/2 t  C5 + 3 C8 t,
```

[2] Heck A 1993 *Introduction to Maple* (New York: Springer-Verlag) pp. 415–416.

```
V2_2(t) = -t C5 + C8, V3_4(t) = -C5,

V3_5(t) = C3, V3_2(t, x) = -x C5 + C3,

V2(t, x, u) = x (- t C5 + C8) + C3 t + C4,

V2_1(t, x) = x (- t C5 + C8) + C3 t + C4,

V3(t, x, u) = u ( t C5 + C6) - x C5 + C3,

t C5 - C6 - 2C8 = 0, V3_1(t, x) = t C5 + C6,

                               2
V1_2(t) = C9 - 3/2 t  C5 + 3 C8 t,

V3_3(t) = t C5 + C6, V2_3(t) = C3 t + C4}
```

The third line from the bottom of this output contains a condition that

```
tC5 - C6 - 2 C8 = 0
```

Heck shows that this condition reduces *Maple's* results to the correct form. However, *Maple* did not provide this condition when attempting to reproduce Heck's result. The *Maple* output from these sessions is included at the end of this appendix.

After determining that Maple's automatic routines were not reliable, the same problem was done using `liesymm's` primitive commands. One can see that the Lie derivatives were taken correctly, and that the error occurred during the reduction modulo process. The explanation requires a few specifics. The KdV equation was written with the following closed set of forms

$$\begin{aligned} \alpha &= du\, dt - z\, dx\, dt \\ \beta &= dz\, dt - p\, dx\, dt \\ \gamma &= -du\, dx + zu\, dx\, dt + \epsilon\, dp\, dt. \end{aligned} \tag{B.11}$$

The Lie derivative of α is

$$\pounds_V \alpha = dV^u dt + du\, dV^t - V^z\, dx\, dt - z\, dV^x dt - z\, dx\, dV^t. \tag{B.12}$$

The differential vector components are now sectioned with the relation $dV^{\hat{\imath}} = V^i_{,j}\, dx^{\hat{\jmath}}$ given in equation (3.4). As an example

$$dV^{\hat{t}} = V^t_{\ ,t}\, d\hat{t} + V^t_{\ ,x}\, d\hat{x} + V^t_{\ ,u}\, d\hat{u} + V^t_{\ ,z}\, d\hat{z} + V^t_{\ ,p}\, d\hat{p}. \tag{B.13}$$

Expanding (B.12) by this process yields a differential form of rank 2 (the hats are dropped for clarity)

$$
\begin{aligned}
\pounds_V \alpha = {} & \left[V^u{}_{,x} - V^z - z(V^x{}_{,x} + V^t{}_{,t})\right] dx\,dt + (V^u{}_{,u} + V^t{}_{,t} - zV^x{}_{,u})\,du\,dt \\
& + (V^u{}_{,z} - zV^x{}_{,z})\,dz\,dt + (V^u{}_{,p} - zV^x{}_{,p})\,dp\,dt + (V^t{}_{,x} + zV^t{}_{,u})\,du\,dx \\
& + zV^t{}_{,z}\,dz\,dx + zV^t{}_{,p}\,dp\,dx + V^t{}_{,z}\,du\,dz + V^t{}_{,p}\,du\,dp.
\end{aligned} \tag{B.14}
$$

The reduction modulo process expresses the dependence of u on (x, t) by substituting $du = u_t dt + u_x dx$ into all 2-forms containing du. Then $du\,dt = z\,dx\,dt$ and $dz\,dt = p\,dx\,dt$. No jet variable corresponds to $u_{,t}$, however an expression for $du\,dx$ is found by solving the annulled form γ, $du\,dx = zu\,dx\,dt + dp\,dt$. After expanding and recombining, the coefficient of each form is set equal to zero, yielding the differential equations for $\boldsymbol{V}$.

$$
\begin{aligned}
& V^t{}_{,p} = 0 \\
& V^t{}_{,z} = 0 \\
& V^u{}_{,p} - zV^x{}_{,p} + \epsilon(V^t{}_{,x} + zV^t{}_{,u}) = 0 \\
& V^u{}_{,x} - V^z - z(V^x{}_{,x} + V^t{}_{,t}) + z(V^u{}_{,u} + V^t{}_{,t} - zV^x{}_{,u}) + p(V^u{}_{,z} - zV^x{}_{,z}) \\
& \qquad + zu(V^t{}_{,x} + zV^t{}_{,u}) = 0.
\end{aligned} \tag{B.15}
$$

Perhaps *Maple* does not recognize the type of substitutions required for $du\,dx$, and merely sets the coefficient of $du\,dx$ equal to zero. This was the best guess during the testing period; however, I did not have the time to locate the precise error. The perceived difficulties with the reduction modulo command `&mod` led to writing the Dtools procedures using multipliers.

Maple produced this output when solving the Korteweg-de Vries equation for the three cases listed above. A couple lines have been reformatted to fit within this paper's margins, and the `"bytes used"` message lines have been eliminated to conserve space. This *Maple* Release 2 session was run on IBM workstations at the University of Utah, because *Maple* has since been upgraded to Release 3 on all BYU machines.

```
    |\^/|     Maple V Release 2 (University of Utah)
._|\|   |/|_. Copyright (c) 1981-1993 by the University of Waterloo.
 \  MAPLE  /  All rights reserved. Maple and Maple V are registered
 <____ ____>  trademarks of Waterloo Maple Software.
```

```
      |       Type ? for help.
> with(liesymm):
> a := Diff(u(t,x),t) + u(t,x)*Diff(u(t,x),x) + k*Diff(u(t,x),x$3);
         /  d         \           /  d         \
    a := |---- u(t, x)| + u(t, x) |---- u(t, x)| + k Diff(u(t, x), x, x, x)
         \ dt         /           \ dx         /

> ad := determine(a,V,u(t,x),w):
bytes used=1000156, alloc=786288, time=1.18
bytes used=28006136, alloc=1572576, time=57.20
bytes used=29006268, alloc=1572576, time=59.15
 as := {} &where {V2_1(t, x) = x (C7 + t C3) + C6 + C4 t,

     V2(t, x, u) = x (C7 + t C3) + C6 + C4 t,

                                   2
     V1(t, x, u) = C5 + 3/4 t  C3 - 3/2 (- 2 C7 - t C3) t,

     V3(t, x, u) = u (- 2 t C3 - 2 C7) + x C3 + C4,

     V3_1(t, x) = - 2 t C3 - 2 C7, V3_3(t) = - 2 t C3 - 2 C7,

                               2
     V1_2(t) = C5 + 3/4 t  C3 - 3/2 (- 2 C7 - t C3) t,

     V3_2(t, x) = x C3 + C4,

     V2_3(t) = C6 + C4 t, V2_2(t) = C7 + t C3, V3_4(x) = x C3 + C4}

> b := Diff(u(t,x),t) + u(t,x)*Diff(u(t,x),x) + Diff(u(t,x),x$3);
          /  d         \           /  d         \
     b := |---- u(t, x)| + u(t, x) |---- u(t, x)| + Diff(u(t, x), x, x, x)
          \ dt         /           \ dx         /

> bd := determine(b,V,u(t,x),w):
bytes used=30006400, alloc=1572576, time=60.92
bytes used=58014160, alloc=1769148, time=118.61
 bs := {} &where {V3_5(0) = C1, V2_3(0) = C6, V3_3(0) = C8,

     V3(0, x, u) = u C8 + x C3 + C1, V3_1(0, x) = C8, V3_4(0) = C3,

     V3_2(0, x) = x C3 + C1, V1(0, x, u) = C9, V2_2(0) = - 1/2 C8,

     V2(0, x, u) = - 1/2 x C8 + C6, V2_1(0, x) = - 1/2 x C8 + C6,

     V1_2(0) = C9}

> c := Diff(u(t,x),t) + u(t,x)*Diff(u(t,x),x) + 12*Diff(u(t,x),x$3);
         /  d         \           /  d         \
    c := |---- u(t, x)| + u(t, x) |---- u(t, x)|
         \ dt         /           \ dx         /

          + 12 Diff(u(t, x), x, x, x)


> cd := determine(c,V,u(t,x),w):
```

```
bytes used=59014292, alloc=1769148, time=120.53
bytes used=88019860, alloc=1834672, time=180.85
                                                         2
 cs := {} &where {V3_5(t) = C2, V1_2(t) = C9 + 3/4 t  C3 - 3/2 C8 t,

                                   2
     V1(t, x, u) = C9 + 3/4 t  C3 - 3/2 C8 t,

     V2_1(t, x) = x (1/2 t C3 - 1/2 C8) + C2 t + C6,

     V2(t, x, u) = x (1/2 t C3 - 1/2 C8) + C2 t + C6,

     V2_2(t) = 1/2 t C3 - 1/2 C8, V3_3(t) = - t C3 + C8,

     V2_3(t) = C2 t + C6,

     V3(t, x, u) = u (- t C3 + C8) + x C3 + C2, V3_1(t, x) = - t C3 + C8,

     V3_2(t, x) = x C3 + C2, V3_4(t) = C3}

> quit;
bytes used=88649276, alloc=1834672, time=181.97
```