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P 387N

Project 2

- This code solves the classical wave equation in one dimension,

$$Q_{,tt} - Q_{,xx} = 0.$$

  The wave velocity is set to unity ($v = 1$). The wave has an initial velocity.

- Two boundary conditions have been implemented especially for this project, (1) periodic boundary conditions and (2) radiation conditions. The periodic boundary conditions map the right grid boundary into the left boundary and vice versa. The radiation condition is:

$$Q_{,t} - Q_{,x} = 0 \qquad \text{at left boundary}$$

$$Q_{,t} + Q_{,x} = 0 \qquad \text{at right boundary.}$$

- The initial data is a gaussian pulse with unit velocity in the positive direction. The analytic solution for this wave is given by the equation

$$Q(t,x) = e^{-100(x-t)^2}$$

  The initial data was set from the above equation with $t = 0$. The initial condition was the time derivative of the analytic solution,

$$\frac{\partial Q}{\partial t} = 200xe^{-100(x-t)^2}$$

- The finite difference approximation to the wave equation is the explicit leap frog scheme, which is based on centered second derivatives. The fundamental FD equation is

$$Q_j^{n+1} = 2Q_j^n - Q_j^{n-1} + \gamma^2(Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n)$$

  where $\gamma = v\Delta t/\Delta x$. This scheme is accurate to order $O[(\Delta t)^2, (\Delta x)^2]$. At the boundaries the centered spatial derivitive is substituted with either backward or forward derivitives, also of order $O[(\Delta x)^2]$.

- The leap frog method requires data at two time levels initally. As mentioned above, the data for the first time level were generated from the analytic equation. The data for the second time step were calculated from this Taylor expansion:

$$Q^1 = Q^0 + Q^0_{,t}\Delta t + \frac{1}{2}Q^0_{,tt}\Delta t^2.$$

  The wave equation makes the identity $Q_{,tt} = Q_{,xx}$, and $Q_{,tt}$ was found by actually calculating $Q_{,xx}$ on the initial data.

- The accuracy of these simulations was tested in two ways. First the convergance factor was calculated by comparing numerical solutions on three different grids. The convergence factor is

$$c = \frac{\|Q_{4h} - Q_{2h}\|}{\|Q_{2h} - Q_h\|}$$

where Q_a represents the numerical solution at a fixed time using a grid where $\Delta x = a$. The convergence factor is plotted for both boundary conditions. The fact that $c = 4$ shows that the scheme is truely second order. The convergence factor is flat for the periodic conditions as is expected. When using the radiation boundary conditions, the convergence factor has a jump as the wave passes through the boundary. Apparently this "increase in accuracy" results from the boundary conditions imposing only one allowed solution.

The second way to test accuracy is to compare the numerical solution with the exact solution. Luckily an exact solution is easily generated for the classical wave equation from the finite difference code. The finite difference scheme generates the exact solution when the Courant parameter is set to unity, $CFL = 1$. Several exact solutions were generated by from the code, and then compared to other runs with $CFL = 0.9$. The convergance factors calculated from the exact solution were identical with those calculated from the previous method. I have included plots only of the convergence factors described in the previous paragraph.

- The plots were generated by a grid with 500 points distributed over the range $[-1, 1]$.