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Project 1

- This code solves the classical wave equation in one dimension,

$$Q_{,tt} - Q_{,xx} = 0.$$

  The wave velocity is set to unity ($v = 1$).
- The radiation boundary condition is used. This condition requires:

$$Q_{,t} - Q_{,x} = 0 \qquad \text{at left boundary}$$

$$Q_{,t} + Q_{,x} = 0 \qquad \text{at right boundary.}$$

- The initial condition is

$$\frac{\partial}{\partial t} Q(t = 0) = 0$$

- The initial data is given by

$$Q(t = 0) = e^{-100x^2}$$

- The finite difference approximation to the wave equation is the explicit leap frog scheme, which is based on centered second derivatives. The fundamental FD equation is

$$Q_j^{n+1} = 2Q_j^n - Q_j^{n-1} + \gamma^2 (Q_{j+1}^n - 2Q_j^n + Q_{j-1}^n)$$

  where $\gamma = v\Delta t/\Delta x$. This scheme is accurate to order $O[(\Delta t)^2, (\Delta x)^2]$. At the boundaries the centered spatial derivitive is substituted with either backward or forward derivitives, also of order $O[(\Delta x)^2]$.
- The leap frog method requires data at two time levels initally. My program calculates $Q(t = 0)$ as mentioned above. The data for $Q(t = \Delta t)$ is also calculated from the known analytic solution

$$Q(t) = \frac{1}{2} e^{100(x+t)^2} + \frac{1}{2} e^{100(x-t)^2}$$

  I consider the use of this equation somewhat a blot on my method. In the next generation (Project 2) I will calculate the first time level by the equation

$$Q^1 = Q^0 + Q^0_{,t} \Delta t + \frac{1}{2} Q^0_{,tt} \Delta t^2 .$$

  The wave allows the substitution $Q_{,tt} = Q_{,xx}$. Thus the first time step can be calculated by knowing the second spatial derivative of the initial data. This can all be done numerically. ($Q^0_{,t}$ must be given as an initial condition.)

- The von Neumann stability analysis requires that

$$\gamma \leq 1$$

for this FD scheme to be stable. Accordingly, I set $\gamma = 0.9$ and calculated the time steps for the programs by $\Delta t = \gamma \Delta x$.

- As noted above, the leap frog scheme is theoretically of order $O[(\Delta t)^2, (\Delta x)^2]$. This can be tested by calculating the convergance factor. Let h represent some fixed grid spacing Δx, then the convergance factor is given by

$$c = \frac{\|Q_{4h} - Q_{2h}\|}{\|Q_{2h} - Q_h\|}$$

where Q_a represents the solution to Q at a fixed time calculated on a grid with spacing $\Delta x = a$. The convergance factor as a function of time is plotted on a graph linked to this web site. The convergance factor is approximately four, but fluctuates a little as the waves leave the grid. A convergance factor of four using an l_2 norm implies that the method truely has second order accuracy.

- The numerical grid contained 2000 points distributed over the closed domain $[-1, 1]$.