Spin or Pin?

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(August 20, 1997)

Abstract

The classification of fermions remains an open question. If they are classified by their behavior under transformations of spacetime by the full Lorentz group – rather than by the orientation preserving Lorentz group – then fermions can be assigned four, rather than two, different parities. The reason is that there is, up to isomorphism, only one Spin group which double covers the orientation preserving Lorentz group, but that there are two Pin groups which cover the full Lorentz group. It has been shown that the vacuum expectation value of the fermionic current defined on a topologically nontrivial configuration space by one Pin group can be totally different from the vacuum expectation value defined by the other one (DeWitt-Morette and DeWitt, 1990). Only one of the two Pin groups can act on the Majorana neutrino emitted and reabsorbed in the process of neutrinoless double beta decay. It is shown that there are no straightforward experiments for assigning a Pin group to a fermion, but there are many areas of physics where parity, time reversal, and

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charge conjugation play a role and may give indirect evidence for making the appropriate assignments. Using the Pin groups, the authors clarify concepts and properties of space and time reversal, charge conjugation, and adjoint (s)pinors necessary for determining a particle parity. The article includes also a semi-historical introduction and a partial bibliography of related literature.

Classification codes: PACS 11.30.Er; MSC 22 E 99

Key Words: Full Lorentz group, Pin group, Intrinsic parity, Time reversal, CPT, Majorana particle.

I. THE CONCEPT OF INTRINSIC PARITY OF A SPIN-1/2 PARTICLE

A. Semi-historical summary.

Racah, in 1937, and Yang and Tiomno, in 1950, pointed out that under a space inversion four different transformations for fields of spin-1/2 are possible. Yang and Tiomno (1950) added, "The types of transformation properties to which the various known spin-1/2 fields belong are physical observables and could in principle be determined experimentally from their mutual interactions and their interactions with fields of integral spin."

Under the impact of superselection rules (Wick *et al.*, 1952), the discovery of parity violation (Lee and Yang, 1956; Wu, 1957), and the success of the Standard Model (Weinberg, 1967; Salam, 1968 and 1964), the Yang and Tiomno paper fell by the wayside; its goal was rendered obsolete. Nevertheless the fact remains that there are four different kinds of spin-1/2 particles (more appropriately called, nowadays, pin-1/2 particles). A careful analysis of parity suggests, as we shall see, a better terminology; namely the spin group is a subgroup of the pin group, just as $SO(n)$ is a subgroup of $O(n)$. Therefore we should write "the fact is that there are four different kinds of pin 1/2 particles". Why has this fact, noted already in 1937, been largely ignored?

1. The impact of superselection rules

In 1952, Wick, Wightman, and Wigner discussed the limitations of the concept of intrinsic parity of elementary particles that follow from superselection rules—i.e., from restrictions on the nature and scope of possible measurements. These restrictions cast a shadow on the possibility of identifying Dirac fields by their transformation properties under space inversion. But it has been shown (DeWitt-Morette and DeWitt, 1990) that the four choices of Pin structure **are** observable—admittedly in a gedanken experiment, but a gedanken experiment is sufficient to rule out a superselection rule. Moreover, as pointed out by Weinberg (1995) "the issue of superselection rules is a bit of a red herring; it may or may not be possible to prepare physical systems in arbitrary superpositions of states, but one cannot settle the question by reference to symmetry principles, because whatever one thinks the symmetry group of nature may be, there is always another group whose consequences are identical except for the absence of superselection rules."

Remark

One cannot recall the work of Wick, Wightman, and Wigner (1952) without mentioning its refutation by Y. Aharanov and L. Susskind (1967). Wick, Wightman, and Wigner suggest that certain phase interferences cannot be measured. More precisely, they assert that one can decompose the Hilbert space of the universe into orthogonal subspaces $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$ such that there are no observables which contain matrix elements between any pair of those subspaces, thus making the relative phases between components of the state vector in different subspaces $\mathcal{H}_i$ and $\mathcal{H}_j (i \neq j)$ irrelevant. Aharanov and Susskind refuted Wick et al.'s claim that there is a superselection rule between states of different electric charges. They showed that it is possible to construct a coherent superposition of states with two different electric charges. Therefore the relative phase between components of the state vector in subspaces $\mathcal{H}_i$ and $\mathcal{H}_j (i \neq j)$, corresponding to charges Q_i and Q_j respectively, is relevant. Indirectly the Aharanov-Susskind paper suggested to us the experiment analyzed in section III.A.

2. The impact of parity violation

The goal of the Yang-Tiomno paper was the assignment of parity to the then-known spin 1/2-particles, which would explain and predict forbidden transitions, assuming a universal Fermi (four fermion) interaction and parity conservation. Once it was established that parity is not conserved in weak interactions, the Yang-Tiomno approach to explaining forbidden transitions was no longer justified, but failure of this approach does not mean that parity cannot be assigned to (s)pin-1/2 particles. Indeed, saying that an interaction between particles does not conserve parity implies that the parity of the initial and final states can be compared. If there were no interaction conserving parity, the concept of parity might be less interesting, but such is not the case.

If one takes seriously the correspondence between particles and irreducible representations of the Poincaré group, one *has to* consider the possibility of four different kinds of (s)pin-1/2 particles characterized by their space reversal transformations.

In particle physics, one considers only two parities. So far, it has been unnecessary to consider four different kinds of particles with (s)pin-1/2 labelled by the four different possible parities. But the discovery of new particles could change this picture, or some discussions could be simplified by using the four *existing* parities. For example the analysis of parity in complex situations occuring in atomic and molecular physics may even require the use of four parities. As long as there are unexplained phenomena in the physics of (s)pin-1/2 particles (e.g. solar neutrinos, albeit a farfetched example) one should keep in mind the fact that the full Lorentz group implies four different parities for (s)pin-1/2 particles.

3. The impact of the Standard Model

In the Standard Model, the spin-1/2 particles are chiral particles defined by the Weyl representation and one might think that the concept of parity does not apply to a chiral particle. Stated in other words, since a left particle becomes a right particle under space inversion, how does one define its parity?

In order to answer this question, as well as to explain statements made in the semi-historical introduction, we need to define the Pin group.

B. Mathematical analysis

Let $O(3,1)$ be the Lorentz group of transformations of $(\mathbf{R}^4, \eta)$ which leaves invariant the quadratic form $\eta_{\alpha\beta}x^\alpha x^\beta$ where

$$(\eta_{\alpha\beta}) := \mathrm{diag}(1,1,1,-1) \qquad ; \tag{1.1}$$

and let $O(1,3)$ be the Lorentz group of transformations of $(\mathbf{R}^4, \hat{\eta})$ where

$$\hat{\eta}_{\alpha\beta} := \mathrm{diag}(1,-1,-1,-1). \tag{1.2}$$

The groups $O(3,1)$ and $O(1,3)$ are isomorphic, but we shall still use different symbols for their elements because $\left[L^{\beta}{}_{\alpha}\right] \in O(3,1)$ is not identical to $\left[\hat{L}^{\beta}{}_{\alpha}\right] \in O(1,3)$.

Although $O(3,1)$ and $O(1,3)$ are isomorphic, invariance of the Dirac equation under the Lorentz transformations $O(3,1)$ and $O(1,3)$ implies the existence of two *nonisomorphic* groups, called Pin (3,1) and Pin (1,3) respectively (see for instance Choquet-Bruhat and DeWitt-Morette 1989). On the other hand, invariance of the Dirac equation under only the orientation preserving subgroups $SO(3,1)$ and $SO(1,3)$ requires only the existence of two *isomorphic* groups, called Spin (3,1) and Spin (1,3). Originally no distinction was made between the Spin groups and the Pin groups – a fact which has led to confusion.

We shall use the following notation:

$Spin^{\uparrow}(3,1)$	double covers	Proper Lorentz group	$SO^{\uparrow}(3,1)$
$Spin(3,1)$	double covers	Orientation preserving Lorentz group	$SO(3,1)$
$Pin^{\uparrow}(3,1)$	double covers	Orthochronous Lorentz group	$O^{\uparrow}(3,1)$
$Pin(3,1)$	double covers	Full Lorentz group	$O(3,1)$

and similar notations for $O(1,3)$, $Pin(1,3)$, and their subgroups. A Ψ-particle will refer to a representation of the $Pin(3,1)$ group and a $\hat{\Psi}$-particle will refer to a representation of the $Pin(1,3)$ group.

In brief

$SO(3,1)$	is a subgroup of	$O(3,1)$
$Spin(3,1)$	is a subgroup of	$Pin(3,1)$

FIGURES

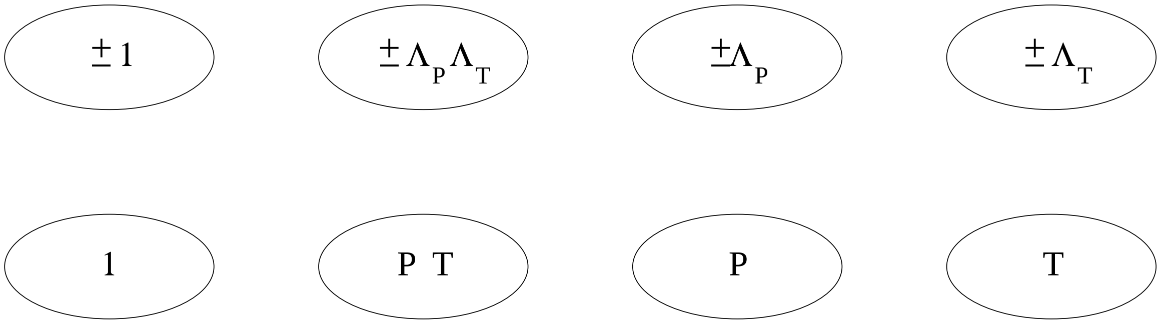

FIG. 1(a). On the bottom row, the four components of the Lorentz group O(3,1) and, on the top row, their double covers Pin(3,1). Each component of O(3,1) is labeled by one of its elements; for instance $P = diag(-1,-1,-1,1)$ is a space inversion and $T = diag(,1,1,-1)$ is a time reversal. In each component of Pin(3,1) we give the two elements which cover the given element of O(3,1). For instance, $\pm\Lambda_P$ satisfies the equations $\Lambda_p \gamma_\alpha \Lambda_P^{-1} = \gamma_\beta P^\beta{}_\alpha$.

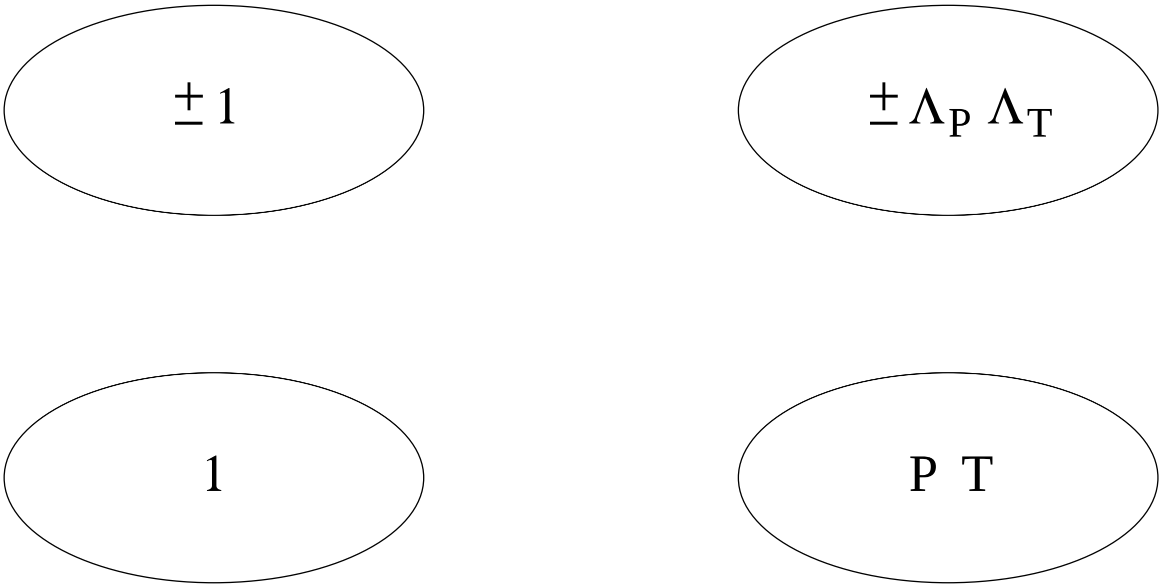

FIG. 1(b). The two components of the orientation preserving Lorentz group SO(3,1)and their double covers Spin(3,1). The expression "orientation preserving" to label the Lorentz matrices of determinant one is commonly used, but literally meaningless. Discussion of parity and time reversal can only be carried out in the framework of the full Lorentz group. Whereas Spin(3,1) and Spin(1,3) are isomorphic, Pin(3,1) and Pin(1,3) are not isomorphic and discussions of properties of fermions under parity and ime reversal can only be carried out in the framework of a Pin group. The question is "which Pin group ?"

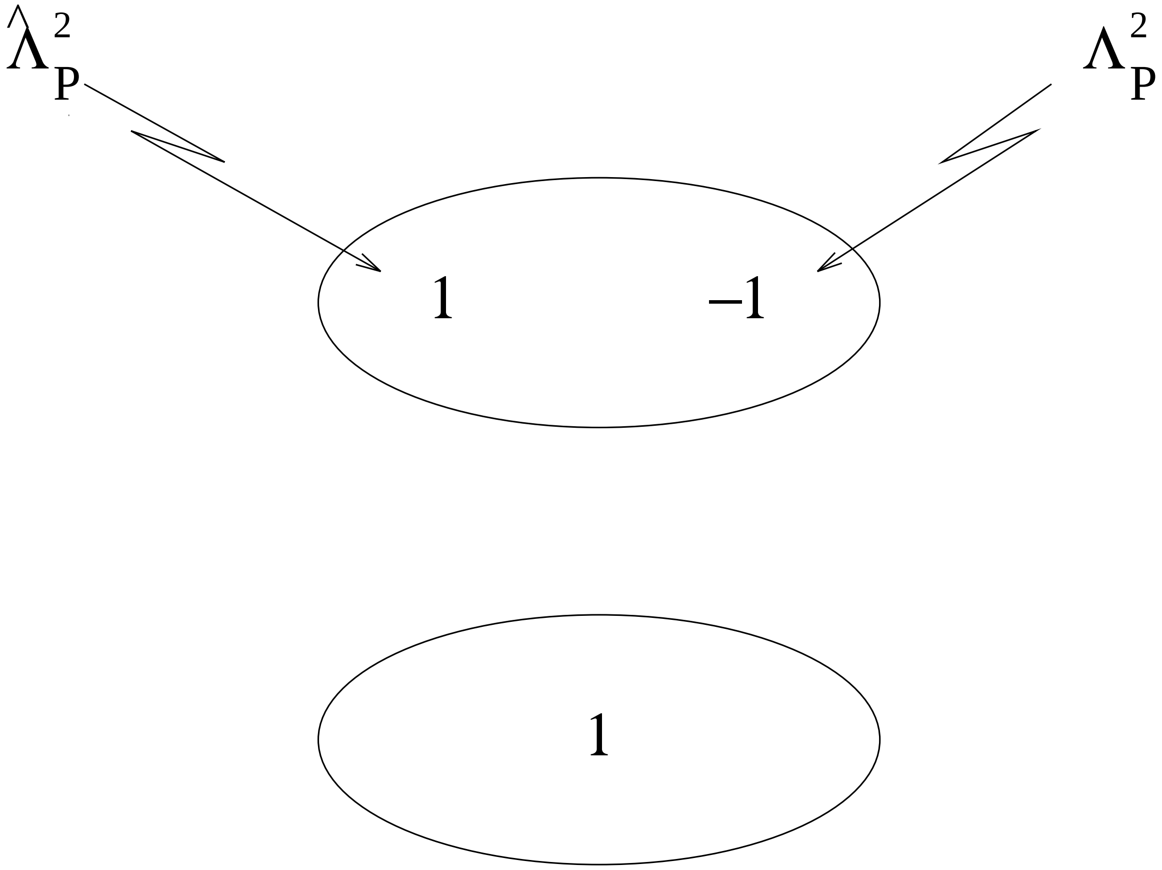

FIG. 1(c). The proper Lorentz group SO$^+$(3,1) and its double cover Spin$^+$(3,1). If Λ_P belongs to Pin(3,1) then $\Lambda_P^2 = -1$; if $\hat{\Lambda}_P$ belongs to Pin(3,1) then $\hat{\Lambda}_P^2 = 1$. To appreciate the difference between the to Pin groups, one can construct the multiplication tables of the four element groups $(\pm 1, \pm\hat{\Lambda}_p)$ and $(\pm 1, \pm\Lambda_P)$; the former is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, the latter to $\mathbb{Z}_4$, where $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are the additive goup of integers modulo 2 and modulo 4 respectively.

The origin of the word Pin is unknown; it is obviously a joke: Pin (3,1) comes from Spin (3,1) as $O(3,1)$ comes from $SO(3,1)$. The joke has been attributed to Serre by Atiyah, Bott and Shapiro (1964), but Serre denied it in a private conversation. It is a very useful word because Spin (3,1) and Spin (1,3) are isomorphic whereas Pin (3,1) and Pin (1,3) are not isomorphic, and this is the reason for the possible existence of four different particles.

The Pin group Pin (3,1) is by definition a double cover of $O(3,1)$, which consists of elements Λ such that

$$\Lambda_L \gamma_\alpha \Lambda_L^{-1} = \gamma_\beta L^\beta{}_\alpha; \qquad \text{equivalently} \;\; \Lambda_L \gamma^\alpha \Lambda_L^{-1} = (L^{-1})^\alpha{}_\beta \gamma^\beta;$$

here

$$\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}\mathbb{1}; \qquad (\eta_{\alpha\beta}) = \mathrm{diag}(1,1,1,-1); \qquad \Lambda_L\Lambda_L^\tau = \pm 1. \tag{1.3}$$

where τ is defined by $(\Lambda_1\Lambda_2)^\tau = (\Lambda_2)^\tau(\Lambda_1)^\tau$ and by

$$(\gamma_\alpha{}^A{}_B)^\tau = (\gamma_\alpha{}^B{}_A)^* .$$

On the other hand, the Pin group $Pin(1,3)$ is a double cover of $O(1,3)$ which consists of elements $\hat{\Lambda}$ such that

$$\hat{\Lambda}_L\hat{\gamma}_\alpha\hat{\Lambda}_L^{-1} = \hat{\gamma}_\beta\hat{L}^\beta{}_\alpha;$$

$$\{\hat{\gamma}_\alpha, \hat{\gamma}_\beta\} = 2\hat{\eta}_{\alpha\beta}\mathbb{1}; \qquad (\hat{\eta}_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1); \qquad \hat{\Lambda}_L\hat{\Lambda}_L^\tau = \pm 1. \tag{1.4}$$

Spinor and pinor fields are sections of vector bundles defined on spacetime; the Spin group and Pin group act on the spinor and pinor fields, respectively. These group actions tell how the components of a field transform under a space inversion in spacetime: they assign a parity to the field.

Under a Lorentz transformation $[L^\alpha{}_\beta] \in O(3,1)$ of spacetime a pinor field transforms according to

$$\Psi'\left(L^\alpha{}_\beta\, x^\beta\right) := \Lambda_L\Psi\left(x^\alpha\right), \tag{1.5}$$

where Λ_L is one of the two elements defined by (1.3). If the Lorentz transformation is the space inversion,

$$\left(P^\alpha_{\ \beta}\right) = \mathrm{diag}(-1,-1,-1,1), \tag{1.6}$$

then $\Lambda_P = \pm\gamma_0$ and for $\Lambda_P \in Pin(3,1)$,

$$\Lambda_P^2 = -\mathbb{1}. \tag{1.7}$$

The eigenvalues of Λ_P are $\pm i$, and the parity is said to be $\pm i$.

Under a Lorentz transformation $\left[\hat{L}^\alpha_{\ \beta}\right]$ of spacetime a pinor field $\hat{\Psi} \to \hat{\Psi}'$ such that

$$\hat{\Psi}'\left(\hat{L}^{\alpha}{}_{\beta}\, x^{\beta}\right) := \hat{\Lambda}_L \hat{\Psi}\left(x^{\alpha}\right), \tag{1.8}$$

where $\hat{\Lambda}_L$ is one of the two elements defined by (1.4). If

$$\left(\hat{P}^{\alpha}{}_{\beta}\right) = \mathrm{diag}(1, -1, -1, -1), \tag{1.9}$$

then $\hat{\Lambda}_P = \hat{\gamma}_o$ and for $\hat{\Lambda}_P \in Pin(1,3)$,

$$\hat{\Lambda}_P^2 = \mathbb{1}. \tag{1.10}$$

The eigenvalues of $\hat{\Lambda}_P$ are ± 1 and the parity is said to be ± 1.

Remark

The general cases $Pin(s,t)$ and $Pin(t,s)$ for $t = 1$ and s arbitrary can be found in the paper by C. DeWitt-Morette and S.-J. Gwo (1990). There, in order to give formulae valid both for s odd and s even, the twisted definition of the Pin groups was used, and space reversals reversed the direction of only one axis. Here we restrict our discussion to $s = 3$, and define space reversal by P and $\hat{P}$ which, as usual, reverse the directions of all three axis. Note in particular that, with the twisted definition, and with s reversing the first axis of $Pin(s,t)$, then $\Lambda_S^2 = \mathbb{1}$.

Remark

Pin (3,1) and Pin (1,3) are not the only double covers of the Lorentz group. See section V for the identification of the eight double covers of the Lorentz group.

We are now in a position to rephrase the question raised at the end of Section I.A.3, and to answer it: In the Standard Model, the spin-1/2 particles are chiral particles defined by the Weyl representation which is not a representation of a Pin group but a representation of the Spin group since only even products of the 4×4 gamma matrices can be block diagonalized. Therefore one can ask, "How does one distinguish Pin (3,1) and Pin (1,3) particles in the Standard Model?" The answer can be found in equations (1.7) and (1.10). The squares Λ_P^2 and $\hat{\Lambda}_P^2$ which distinguish Pin (3,1) and Pin (1,3) particles are even products of gamma matrices and respect the block diagonal form of matrices in the Weyl representation.

In particle physics, one chooses one of the two possible Pin groups and considers only the two parities associated with the two possible eigenvalues of the parity operator in the chosen group.

Remark

In his book "The Quantum Theory of Fields Vol. I", S. Weinberg (1995) considers only Pin (3,1) and uses $\Lambda_P^2 = \mathbb{1}$ which is inconsistent with (1.7). But since the author argues (p. 125) that a phase transformation of Λ_P^2 is of no consequence (except for Majorana neutrinos) in the context of his book, he can work with $\Lambda_P^2 = \mathbb{1}$. Such would not be the case, for instance, in nonorientable superstring theories (Carlip and DeWitt-Morette, 1988).

C. The issue nowadays.

It has been mentioned earlier that the four choices of Pin structures are, in principle, observable. Indeed it has been shown by C. DeWitt-Morette and B.S. DeWitt (1990) that they give different vacuum expectation values of Fermi currents on $\mathbf{R} \times (\mathbf{R}\times$ Klein bottle). This calculation is important conceptually, but it is of little interest for an experimentalist—at least for the time being. Moreover, it gives the impression that a topologically nontrivial base space is needed to exhibit the differences between the various Pin structures. It *is* true that on a trivial base space, a Pin-bundle is reducible to a Spin-bundle. This means that on the overlap $U_i \cap U_j$ of two patches of the base manifold, the pinor field $\Psi_j \equiv \Psi\mid_{U_j}$ and the pinor field $\Psi_i \equiv \Psi\mid_{U_i}$ are related by

$$\Psi_j = \Lambda \Psi_i \tag{1.11}$$

with Λ an element of the Spin group. But Ψ can nevertheless be acted upon by the full Pin (3,1) group, and be distinguished from $\hat{\Psi}$.

The purpose of this paper is to clarify the properties of pinor representations and to examine the possibility of finding experimentally to which of the two Pin groups a given (s)pin-1/2 particle belongs. If all the (s)pin-1/2 particles do not belong to the same Pin group, we would obtain a classification of (s)pin-1/2 particles based on the full Lorentz group, finer than the classification based on the proper Lorentz group.

In Section II we relate the unitary time reversal operator Λ_T to the antiunitary time reversal operator Θ, used in quantum physics, and investigate if the Kramers' degeneracy can be used to distinguish Ψ and $\hat{\Psi}$-particles. The operators Λ_T and Θ are related by charge conjugation which is reviewed in Section IV.

In Section III we investigate the possibility of determining Λ_P^2 or $\hat{\Lambda}_P^2$ for electrons.

In Section IV, we construct the charge conjugation operators, both the antiunitary operator acting on the Dirac wave function and the unitary operator acting on the Dirac field operator. *We establish that Majorana neutrinos, if they exist, can only be Ψ particles.*

In Section V, we discuss briefly the eight double covers of the Lorentz group.

In Appendix A we recall the definitions of Dirac and Majorana adjoints and their use in defining charge conjugation.

In Appendix B we establish that a CPT transformation is merely an orientation preserving Lorentz transformation.

In Appendix C we compare definitions and equations used when working with Pin (3,1) and Pin (1,3).

We are only considering the *real* Pin groups, defined by (1.3) and (1.4), and their real or complex representations. Our choice of real Pin groups is based on two arguments. A real Pin group is constructed from a real Clifford algebra, whose linear subspace spanned by the γ_α is isomorphic with real spacetime. Furthermore, experiments have shown that spin-1/2 particles indeed behave under rotations as representations of a double cover of $O(3,1)$ or $O(1,3)$ (see e.g. Rauch *et al.*, 1975 and Werner *et al.*, 1975), and the double covers of $O(3,1)$ and $O(1,3)$ are the real Pin groups Pin (3,1) and Pin (1,3), respectively.

II. TIME REVERSAL AND THE KRAMERS' DEGENERACY

Kramers has shown (see e.g. Sakurai, 1985, p. 281 or Merzbacher, 1970,p. 408; see also Avron *et al.*, 1989, p.601) that, given an **antiunitary** operator Θ such that

$$\Theta\Theta^* = -\mathbb{1}$$

and such that it commutes with the hamiltonian H of the system, the eigenstate ($|n\rangle$) of H and the time reversed eigenstate $\Theta(|n\rangle^*)$ are two different states with the same energy. This degeneracy can be removed by adding an interaction which does not commute with Θ.

The proof of Kramers' theorem is as follows. If $|n\rangle$ and $\Theta(|n\rangle^*)$ were the same state, then $\Theta(|n\rangle^*) = e^{i\delta}|n\rangle$. The time reversed of the time reversed state $\Theta(|n\rangle^*)$ is $\Theta\Theta^*|n\rangle = e^{-i\delta}\Theta(|n\rangle^*) = |n\rangle$. Since $\Theta\Theta^* = -\mathbb{1}$, the state $|n\rangle$ and its time reversed state $\Theta(|n\rangle^*)$ cannot be the same state. If Θ commutes with the Hamiltonian, the states $|n\rangle$ and $\Theta(|n\rangle^*)$ are two different states with the same energy.

Can the Kramers' theorem provide a method for distinguishing Ψ-particles (with $\Lambda_T^2 = -\mathbb{1}$) from $\hat{\Psi}$-particles (with $\hat{\Lambda}_T^2 = \mathbb{1}$)? Before answering this question we must establish the relationship between the antiunitary time reversal operator Θ and the unitary time reversal operator Λ_T.

A. Mathematical (alias relativistic, unitary) time reversal and Physical (alias nonrelativistic, antiunitary) time reversal

Maxwell equations read either

$$\begin{cases} \operatorname{div} \vec{H} = 0, \ \operatorname{curl}\vec{E} + \dot{\vec{H}} = 0, \\ \operatorname{div} \vec{E} = \rho, \ \operatorname{curl}\vec{H} - \dot{\vec{E}} = \vec{\jmath}, \end{cases} \tag{2.1}$$

or, if one wants to emphasize their relativistic invariance,

$$\begin{cases} d\mathbf{F} = 0, \\ \delta\mathbf{F} + \mathbf{J} = 0. \end{cases} \tag{2.2}$$

In the context of Eqs. (2.1), one usually considers a time reversal operator Θ_{MR} (short for motion reversal) that does not change the sign of the charge. On the other hand, the covariant current four vector $\mathbf{J}$ in Eqs. (2.2) transforms under a Lorentz transformation $[L^{\alpha}{}_{\beta}] \in O(3,1)$, as follows:

$$J'_{\alpha}\left(L^{\delta}{}_{\gamma}\, x^{\gamma}\right) L^{\alpha}{}_{\beta} = J_{\beta}\left(x^{\delta}\right). \tag{2.3}$$

In particular, if $[L^{\alpha}{}_{\beta}]$ is the time-reversal transformation T, i.e.,

$$[T^{\alpha}{}_{\beta}] = \mathrm{diag}(1,1,1,-1), \tag{2.4}$$

the charge density changes sign:

$$J'_4(\vec{x},-t) = -J_4(\vec{x},t). \tag{2.5}$$

Let $\mathcal{C}$ be the charge operator that changes the sign of the charge, then

$$T = \Theta_{MR}\mathcal{C} = \mathcal{C}\Theta_{MR}. \tag{2.6}$$

A similar relationship exists between the time reversal operator Λ_T acting on a pinor field

$$\Psi'\left(T^{\alpha}{}_{\beta}\, x^{\beta}\right) = \Lambda_T \Psi\left(x^{\alpha}\right), \quad \Lambda_T \Gamma_{\alpha} \Lambda_T^{-1} = \Gamma_{\beta} T^{\beta}{}_{\alpha}, \tag{2.7}$$

where $\{\Gamma_{\alpha}\}$ is a matrix representation of $\{\gamma_{\alpha}\}$, and the antiunitary time reversal operator Θ used in physics

$$\Psi'\left(T^{\alpha}{}_{\beta}\, x^{\beta}\right) = \Theta \Psi^{*}\left(x^{\alpha}\right). \tag{2.8}$$

Indeed, we shall show that

$$\Lambda_T = \Theta \mathcal{D}_{+}^{-1}, \tag{2.9}$$

where the antilinear charge conjugation operators $\mathcal{D}_{+}$ and $\mathcal{D}_{-}$ are defined by

$$\mathcal{D}_{\pm}^{-1} \Gamma_{\alpha} \mathcal{D}_{\pm} = \pm \Gamma_{\alpha}^{*}. \tag{2.10}$$

The proof of Eqs. (2.9) is as follows:

The Dirac equation

$$\left(\Gamma^{\alpha}\left(\partial_{\alpha}+iqA_{\alpha}\right)-m\right)\Psi=0 \tag{2.11}$$

is invariant under antiunitary transformations $\mathcal{A}$ of Ψ induced by Lorentz transformations

$$\Psi\left(L^{\alpha}{}_{\beta}\,x^{\beta}\right)=\mathcal{A}\Psi^{*}\left(x^{\alpha}\right) \tag{2.12}$$

if

$$\mathcal{A}\Gamma^{\alpha *}\mathcal{A}^{-1}=\left(L^{-1}\right)^{\alpha}{}_{\beta}\,\Gamma^{\beta}. \tag{2.13}$$

This condition is equivalent to

$$\pm\left(\mathcal{A}\mathcal{D}_{\pm}^{-1}\right)\Gamma^{\alpha}\left(\mathcal{D}_{\pm}\mathcal{A}^{-1}\right)=\left(L^{-1}\right)^{\alpha}{}_{\beta}\,\Gamma^{\beta}. \tag{2.14}$$

If, in (2.12), we set $L^{\alpha}{}_{\beta}=T^{\alpha}{}_{\beta}$, then $\mathcal{A}=\Theta$, and the above condition is equivalent to

$$\Theta\mathcal{D}_{+}^{-1}=\Lambda_{T}. \qquad \blacksquare \tag{2.15}$$

There are several reasons for introducing the antiunitary operator Θ in 2.8. One of them is the requirement, necessary in a theory free of negative energy states, that the fourth component of the energy-momentum vector does not change sign under time reversal. Both antiunitary time reversal operators Θ_{MR} in 2.6 and Θ in 2.8 are introduced in physics to satisfy positivity requirements – charge or energy, as the case may be. The operators Θ_{MR} and Θ defined respectively by 2.6 and 2.8 can be called physical time reversal operators; and T and Λ_T defined respectively by 2.4 and 2.7 can be called mathematical time reversal operators.

B. Kramers' degeneracy

Kramers' theorem does not apply to Λ_T (although $\Lambda_T^2=-\mathbb{1}$) because Kramers' theorem does not apply to **unitary** time reversal operators.

Therefore we consider the antiunitary operators Θ and $\hat{\Theta}$ corresponding to Λ_T and $\hat{\Lambda}_T$, respectively.

$$\Lambda_T = \Theta D_+^{-1}; \qquad \hat{\Lambda}_T = \hat{\Theta}\hat{D}_-^{-1}$$

A double time reversal is not produced by Θ^2 but by $\Theta\Theta^*$, since, according to the defining equation 2.8,

$$\Psi''\left(x^\alpha\right) = \Theta\left(\Theta\Psi^*\left(x^\alpha\right)\right)^* = \Theta\Theta^*\Psi\left(x_\alpha\right). \tag{2.16}$$

Whereas $\Lambda_T^2 \neq \hat{\Lambda}_T^2$, the double physical time reversal

$$\Theta\Theta^* = \hat{\Theta}\hat{\Theta}^* = -\mathbb{1}. \tag{2.17}$$

Both Ψ- and $\hat{\Psi}$-particles can be used to construct degenerate time-reversed pairs. It follows that Kramers' degeneracy cannot be used to distinguish Ψ-particles and $\hat{\Psi}$-particles.

III. SPACE REVERSAL

In $Pin(3,1)$ and $Pin(1,3)$, two successive parity transformations are given, respectively, by $\Lambda_P^2 = -\mathbb{1}$ and $\hat{\Lambda}_P^2 = \mathbb{1}$. Hence if one could construct an experiment corresponding to Fig. 2, one might be able to differentiate between the two types of pinor particles. A particle field must somehow be split into two fields, one of which is inverted twice while the other is left unaffected. Then the two fields must be brought back together and allowed to interfere. Where $Pin(3,1)$ particles interfere destructively, $Pin(1,3)$ particles will interfere constructively. The transformations $\Lambda_P = \pm\gamma_0$ and $\hat{\Lambda}_P = \pm\hat{\gamma}_0$, corresponding to a single parity transormation, also differ and could be used in place of the two successive parity transformations in this type of experiment. The analysis of the experiment might become more involved, however, since Λ_P and $\hat{\Lambda}_P$ are not proportional to the identity transformation.

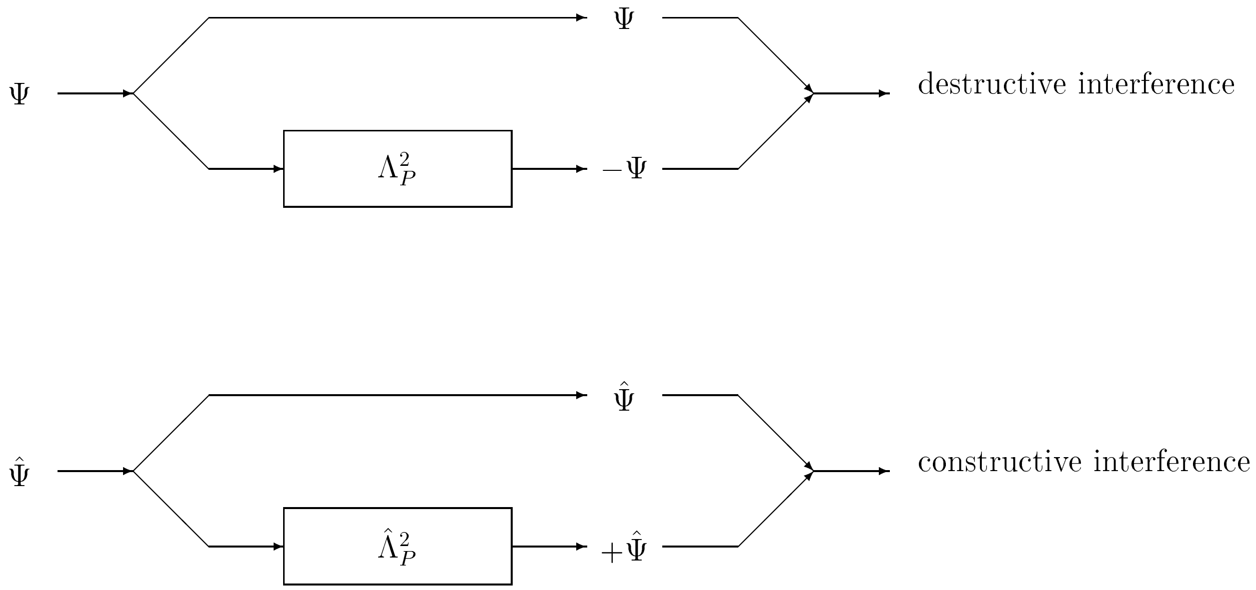

FIG. 2. A type of experiment which should give different results for the two types of pinors.

A. Parity transformation by reversing magnetic fields?

Not knowing how to construct a space reversal apparatus (the boxes in Fig. 2) we considered the following experiment:

A particle beam is split in two parts: One part passes through a magnetic field in the $\hat{\vec{x}}$ direction followed by a magnetic field in the $\hat{\vec{y}}$ direction as in Fig. 3, the other part passes through a magnetic field in the $\hat{\vec{x}}$ direction followed by a magnetic field in the $-\hat{\vec{y}}$ direction. The two parts of the beam are then recombined and interfere. One part of the beam will have passed through a magnetic field pattern which could be obtained from the magnetic field pattern through which the other part of the beam passed by a simple parity transformation $\{\hat{\vec{x}} \longmapsto \hat{\vec{x}}, \hat{\vec{y}} \longmapsto -\hat{\vec{y}}, \hat{\vec{z}}, \longmapsto \hat{\vec{z}}\}$; we thought this would be equivalent to one part of the beam's having undergone a parity transformation relative to the other part of the beam, and would achieve the purpose of the type of experiment shown in Fig. 2.

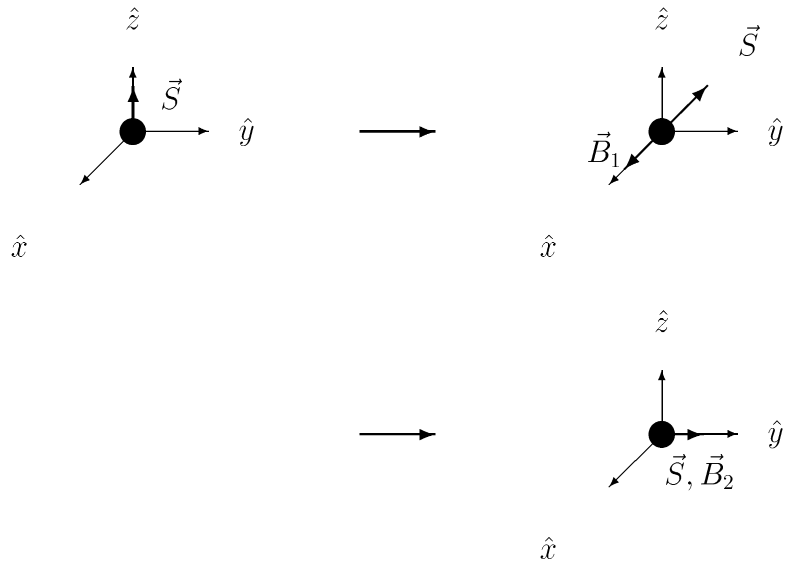

FIG. 3. A z-polarized election is placed for a while in a magnetic field $\vec{B}_1$ along the x-axis causing its spin to precess about the x-axis; later on a magnetic field $\vec{B}_2$ in the y-direction is switched on. One could also consider an electrically neutral spin 1/2 particle in a potential $\mu \vec{s} \cdot \vec{B} = \mu \epsilon_{ijk} \sigma^{ij} B^k \quad (i, j, k, \epsilon\{1, 2, 3\})$ where μ is the gyromagnetic ratio and σ^{ij} the spin angular momentum operator.

An explicit calculation by Eric Kramer (1991) showed that the interference of the two parts of the beam having experienced the different magnetic field configuations $(\vec{B}_1, \vec{B}_2)$ and $(\vec{B}_1, -\vec{B}_2)$ is the same for a Pin (3,1) beam and for a Pin (1,3) beam. With the confidence based on an explicit nontrivial calculation we can give the reason why the proposed set up does not test parity. "Obvious" would say some; "Obvious arguments are not reliable to settle subtle issues" is the obvious retort.

i) The explicit calculation consists in comparing the transition amplitudes for Pin (3,1) and Pin (1,3) electrons moving under the conditions described in Fig. 3. The system evolves according to the Dirac equation which reads

$$(\Gamma^\alpha(\partial_\alpha + iqA_\alpha) - m)\psi = 0 \quad \text{for} \quad \text{Pin}(3,1)\text{particles}$$

$$(\hat{\Gamma}^\alpha(\partial_\alpha + iqA_\alpha) + im)\hat{\psi} = 0 \quad \text{for} \quad \text{Pin}(1,3)\text{particles}$$

The $\hat{\Gamma}^\alpha$ may be represented as matrices equal to $-i\Gamma^\alpha$; the equations (albeit written differently) represent the same evolution. Let the initial states $|\psi(t_i)>$ and $|\hat{\psi}(t_i)>$ be given respectively by

$$|\psi(t_i)>=\int d^3p\Sigma_{\alpha\beta}c_{\alpha\beta}(p)|p,\alpha,\beta>$$
$$|\hat{\psi}(t_i)>=\int d^3p\Sigma_{\alpha\beta}\hat{c}_{\alpha\beta}(p)|\widehat{\vec{p},\alpha,\beta}>$$

where $|\vec{p},\alpha,\beta>$ are Pin (3,1) electron eigenstates with $p_\mu=\alpha(-\vec{p},\sqrt{m^2+|\vec{p}|^2})$ and spin $\beta/2$ in the $\vec{p}$-direction, and $|\widehat{\vec{p},\alpha,\beta}>$ are eigenstates for Pin (1,3) electrons. These two intitial states represent the same physical state if and only if $c_{\alpha\beta}(\vec{p})=\hat{c}_{\alpha\beta}(\vec{p})$ for all $\{\vec{p},\alpha,\beta\}$. Two equivalent initial states, under the same evolution, remains equivalent throughout all time. Both parts of the beam in the Pin (3,1) case evolve exactly the same way as their respective counterparts in the Pin (1,3) case, the interference patterns produced in either case are identical. The same will hold true for any electromagnetic configuration.

B. Parity transformation by reflection?

Since the previous experiment turned out not to give a parity transformation it was suggested by several colleagues to Eric Kramer that he should look at the interference between the two parts of a particle beam passing through some medium M as shown in Fig. 4. The part which follows path 1 is transmitted directly through the medium, whereas the part which follows path 2 is reflected twice before passing through. We were told that such an experiment could be achieved by passing neutrons through a magnetic crystal.

Although the idea of reflecting might bring to mind the idea of making a parity transformation, the reflections involved here have nothing to do with the parity transformation Λ_P (or $\hat{\Lambda}_P$). In other words, although the neutrons following path 2 are reflected twice, they do not undergo any parity transformations as called for in Fig. 2. Hence this setup also fails to realize Fig. 2.

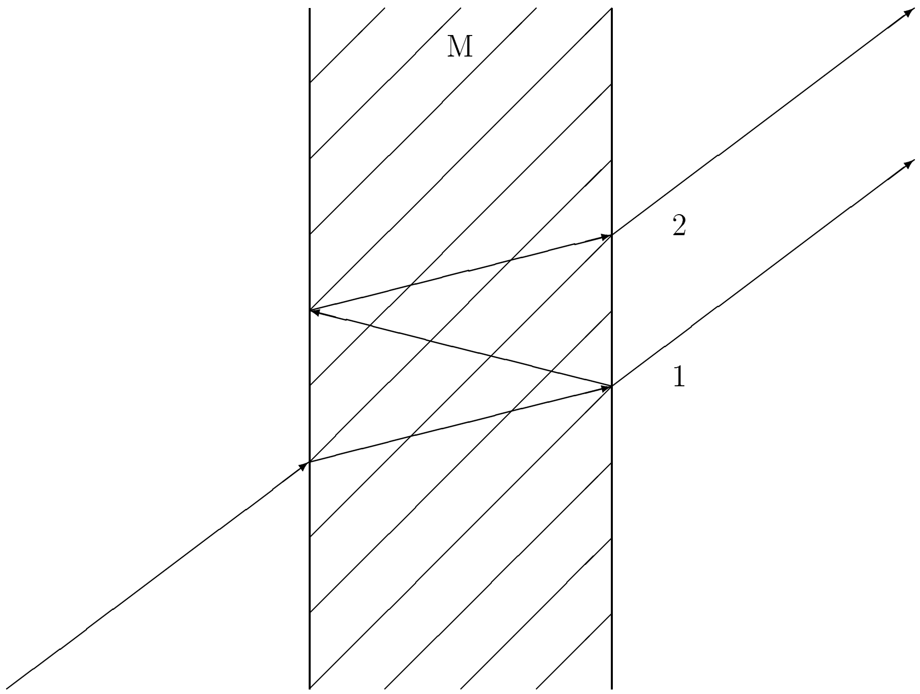

FIG. 4. A reflection experiment intended to find a difference between two types of pinors.

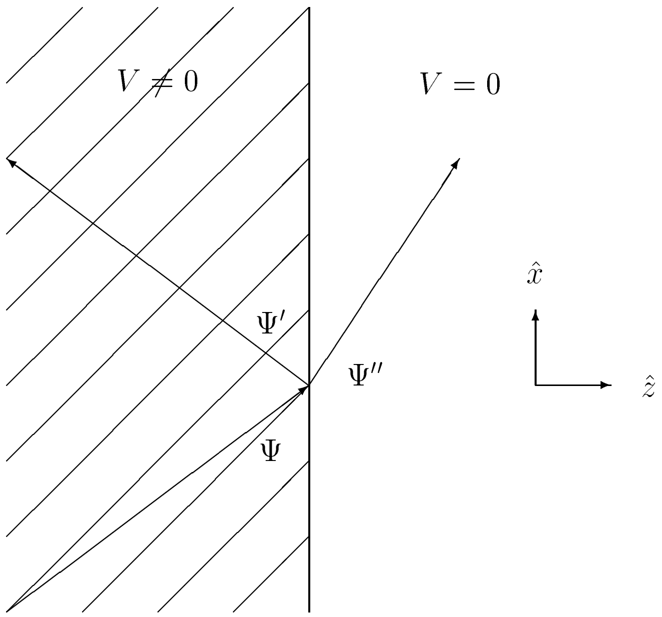

FIG. 5. Reflection and transmission of a pinor particle at an interface.

That reflection at a boundary does not produce any parity transformation is best seen by just solving the problem of reflection and transmission of a plane wave between two media, one with a free-particle Dirac Hamiltonian and one with a Hamiltonian consisting of the free-particle part plus a potential V, as in Fig. 5. The solution of this problem can be found in several books. See, for instance, S. Flügge (1971) for a detailed presentation (in the Dirac representation). In brief, given an incoming Pin (3,1) plane wave of momentum $\vec{p}$

$$\psi = \Sigma_{\alpha,\beta} c_{\alpha,\beta} \exp(iVt) < \vec{x}, t|\vec{p}, \alpha, \beta >, \tag{3.1}$$

requiring the final solution to be continuous at $z = 0$ implies that there must also be a reflected plane wave $\Psi' = \sum_{\alpha\beta} c'_{\alpha\beta} e^{iVt} \langle \vec{x}, t|\vec{p}' \alpha\beta\rangle$ and a transmitted plane wave $\Psi'' = \sum_{\alpha\beta} c''_{\alpha\beta} \langle \vec{x}, t|\vec{p}'' \alpha\beta\rangle$, as shown in Fig. 5. The $\vec{p}', \vec{p}'', c'_{\alpha\beta}$ and $c''_{\alpha\beta}$ can be solved for in terms of $\vec{p}$ and $c_{\alpha\beta}$ from the requirement that the full solution be continuous (i.e. requiring the equality in the limit $z \to 0$ of $\Psi + \Psi'$, which defines the solution for $z < 0$, with Ψ'', which defines the solution for $z > 0$). The boundary condition at $z = 0$ leads to the same equations for getting $(\vec{p}', \vec{p}'', c'_{\alpha\beta}, c''_{\alpha\beta})$ for two equivalent Pin (3,1) and Pin (1,3) incoming plane waves.

In other words a given $\vec{p}$ and $c_{\alpha\beta}$ will imply the same $\{\vec{p}', \vec{p}'', c'_{\alpha\beta}, c''_{\alpha\beta}\}$ in Pin(3,1) as in Pin(1,3), representing the same physical content for each case. Neither Λ_P (nor $\hat{\Lambda}_P$) enters the mathematical description of reflection.

IV. CHARGE CONJUGATION AND MAJORANA PARTICLES

We shall prove that Majorana particles, if they exist, can only be Ψ-particles. A similar conclusion had been reached by Yang and Tiomno (1950), and by Berestetskii, Lifshitz and Pitaevskii (1982), but without any indication that there are two inequivalent Pin groups. To identify Majorana particles as Ψ-particles we need to review the definition of Majorana particles, antiparticles, and charge conjugation (electric and nonelectric charges):

- A particle is said to be Majorana if it is identical with its antiparticle.
- Particles and antiparticles have opposite intrinsic quantum numbers (charge).

Therefore we begin with a review of charge conjugation.

A. Particles and antiparticles

At first charge conjugation meant the electric charge conjugation of the Dirac equation. The fields Ψ^c and Ψ are charge conjugate if they satisfy the charge conjugate equations

$$(\Gamma^\alpha (\partial_a + iqA_\alpha) - m)\,\Psi = 0 \tag{4.1}$$

$$(\Gamma^\alpha (\partial_\alpha - iqA_\alpha) - m)\,\Psi^c = 0. \tag{4.2}$$

Hence

$$\Psi^c = \mathcal{D}_+ \Psi^* \qquad \text{where} \qquad \mathcal{D}_\pm^{-1} \Gamma_\alpha \mathcal{D}_\pm = \pm \Gamma_\alpha^*. \tag{4.3}$$

For massless fields, but for massless fields only, we note that $\mathcal{D}_- \Psi^*$ also defines a charge conjugate field of Ψ with the same energy spectrum as Ψ. In the Dirac representation of the gamma matrices, $\mathcal{D}_+ = \pm\Gamma_2$, $\mathcal{D}_- = \pm\Gamma_0\Gamma_1\Gamma_3$.

Later on the notion of charge conjugate particles has been extended to electrically neutral pairs of particles, and to charges other than electrical—"charge" conjugate pairs being called particles-antiparticles.

$$\textbf{If } \Psi \textbf{ is a spin-1/2 particle, then } \Psi^c := \mathcal{D}_+ \Psi^* \textbf{ is an antiparticle}. \tag{4.4}$$

Nowadays, in the context of quantum field theory, a Dirac field operator $\boldsymbol{\Psi}$ acting on a Fock space of particle and antiparticle states is decomposed into particle and antiparticle plane wave solutions of the Dirac equation,

$$\Psi(x) = u(\vec{p}, s) \exp(ip \cdot x) \qquad\qquad \text{particle} \tag{4.5a}$$

$$\Psi^c(x) = v(\vec{p}, s) \exp(-ip \cdot x) = \mathcal{D}_+ \Psi^*(x) \qquad \text{antiparticle}, \tag{4.5b}$$

as follows:

$$\boldsymbol{\Psi}(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p}, s} \Big(a(\vec{p}, s) u(\vec{p}, s) \exp(ip \cdot x) + b^\dagger(\vec{p}, s) v(\vec{p}, s) \exp(-ip \cdot x) \Big) \tag{4.6}$$

where

$a(\vec{p},s)$ annihilates a particle of momentum $\vec{p}$, spin state s

$b^\dagger(\vec{p},s)$ creates an antiparticle of momentum $\vec{p}$, spin state s.

The charge conjugate operator is

$$\mathbf{\Psi}^c(x) = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p},s} \Big(b(\vec{p},s) u(\vec{p},s) \exp(ip\cdot x) + a^\dagger(\vec{p},s) v\,(\vec{p},s) \exp(-ip\cdot x) \Big) \tag{4.7}$$

where b annihilates an antiparticle and $a^\dagger$ creates a particle

$$\Big[a(\vec{p},s), a^\dagger\,(\vec{p}',s') \Big] = \delta\,(\vec{p}-\vec{p}')\,\delta_{s,s'}; \qquad \Big[b(\vec{p},s), b^\dagger\,(\vec{p}',s') \Big] = \delta\,(\vec{p}-\vec{p}')\,\delta_{s,s'}. \tag{4.8}$$

The charge conjugation operator U_c on Fock space, is by definition the unitary operator on Fock space which exchanges particles and antiparticles

$$\begin{aligned} U_c a(\vec{p},s) U_c^\dagger &= b(\vec{p},s) \\ U_c b(\vec{p},s) U_c^\dagger &= a(\vec{p},s). \end{aligned} \tag{4.9}$$

Hence

$$\mathbf{\Psi}^c = U_c \mathbf{\Psi} U_c^\dagger. \tag{4.10}$$

Since $\mathcal{D}_+\mathcal{D}_+^* = \mathbb{1}$ in $Pin(3,1)$, it follows that

$$\mathbf{\Psi}^c = \mathcal{D}_+ \mathbf{\Psi}^\dagger \tag{4.11}$$

where the dagger denotes hermitian adjoint operator on the Fock space, and complex conjugate in the space spanned by the plane wave solutions u and v.

B. Majorana particles are Pin (3,1) particles

By definition a Majorana wave function Ψ^M, or a Majorana field operator $\mathbf{\Psi}^M$, are such that

$$\Psi^M(x) = \left(\Psi^M\right)^c(x) = \mathcal{D}_+\Psi^{M*}(x) \tag{4.12}$$

$$\begin{aligned}\mathbf{\Psi}^M(x) = \left(\mathbf{\Psi}^M\right)^c &= \mathcal{D}_+\mathbf{\Psi}^{M\dagger}(x) \\ &= \frac{1}{\sqrt{\Omega}}\sum_{\vec{p},s}\left(a(\vec{p},s)u(\vec{p},s)\exp(ip\cdot x) + a^\dagger(\vec{p},s)v(\vec{p},s)\exp(-ip\cdot x)\right).\end{aligned} \tag{4.13}$$

Either definition implies that a Majorana particle is a Ψ-particle, i.e., the Ψ^M-pinors, and u and v-Majorana pinors belong to a representation of $Pin(3,1)$.

The argument of the proof is the same in both cases: To be meaningful, the *defining* condition must remain satisfied under a parity transformation. That is

$$\begin{aligned}\Lambda_P\Psi^M = \left(\Lambda_P\Psi^M\right)^c &= \mathcal{D}_+\left(\Lambda_P\Psi^M\right)^* \\ &= \mathcal{D}_+\Lambda_P^*\mathcal{D}_+^{-1}\mathcal{D}_+\Psi^{M*} \\ &= \mathcal{D}_+\Lambda_P^*\mathcal{D}_+^{-1}\Psi^M.\end{aligned}$$

Hence

$$\Lambda_P = \mathcal{D}_+\Lambda_P^*\mathcal{D}_+^{-1} \tag{4.14}$$

which is precisely the defining equation (2.10) of $\mathcal{D}_+$. ■

On the other hand, the charge conjugate $\hat{\Psi}$-particles are solutions of the charge conjugate equations

$$\left(\hat{\Gamma}^\alpha\left(\partial_\alpha + iqA_\alpha\right) + im\right)\hat{\Psi} = 0$$

$$\left(\hat{\Gamma}^\alpha\left(\partial_\alpha - iqA_\alpha\right) + im\right)\hat{\Psi}^c = 0.$$

Hence

$$\hat{\Psi}^c = \hat{\mathcal{D}}_-\hat{\Psi}^* \qquad \text{where} \qquad \hat{\mathcal{D}}_\pm^{-1}\hat{\Gamma}_\alpha\hat{\mathcal{D}}_\pm = \pm\hat{\Gamma}_\alpha^*.$$

The condition

$$\hat{\Psi}^M = \hat{\Psi}^{Mc} = \hat{\mathcal{D}}_-\hat{\Psi}^*$$

cannot be satisfied since $\hat{\mathcal{D}}_-\hat{\Lambda}_P^*\hat{\mathcal{D}}_-^{-1} = -\hat{\Lambda}_P$.

A similar argument applies to the field operator $\mathbf{\Psi}^M$. Under a parity transformation

$$u(\vec{p}, s) \mapsto \Lambda_P u(\vec{p}, s)$$

$$v(\vec{p}, s) \mapsto \Lambda_P v(\vec{p}, s),$$

$\mathbf{\Psi}^M(x) = \mathbf{\Psi}^{Mc}(x) = \mathcal{D}_+ \mathbf{\Psi}^{M\dagger}(x)$ implies

$$\begin{aligned}
\left(\Lambda_P \mathbf{\Psi}^M\right)^c &= \mathcal{D}_+ \left(\Lambda_P \mathbf{\Psi}^M\right)^\dagger \\
&= \mathcal{D}_+ \Lambda_P^* \mathbf{\Psi}^{M\dagger} \\
&= \mathcal{D}_+ \Lambda_P^* \mathcal{D}_+^{-1} \mathbf{\Psi}^M, \\
&= \Lambda_P \mathbf{\Psi}^M \qquad \text{by the definition of } \mathcal{D}_+.
\end{aligned}$$

In $Pin(1,3)$, on the other hand, one has

$$\hat{\mathbf{\Psi}} = \frac{1}{\sqrt{\Omega}} \sum_{\vec{p},s} \left(\hat{a}(\vec{p}, s)\hat{\Psi}_{\vec{p},s}(x) + \hat{b}^\dagger(\vec{p}, s)\hat{\Psi}^c_{\vec{p},s}(x)\right)$$

and

$$\hat{\mathbf{\Psi}}^c = \hat{\mathcal{D}}_- \hat{\mathbf{\Psi}}^\dagger$$

(since $\hat{\mathcal{D}}_- \hat{\mathcal{D}}_-^* = \mathbb{1}$). Therefore $\hat{\mathbf{\Psi}}(x) = \hat{\mathbf{\Psi}}^c(x) = \hat{\mathcal{D}}_- \hat{\mathbf{\Psi}}^\dagger(x)$ would imply

$$\begin{aligned}
\left(\hat{\Lambda}_P \hat{\mathbf{\Psi}}\right)^c &= \hat{\mathcal{D}}_- \left(\hat{\Lambda}_P \hat{\mathbf{\Psi}}\right)^\dagger \\
&= \hat{\mathcal{D}}_- \hat{\Lambda}_P^* \hat{\mathbf{\Psi}}^\dagger \\
&= \hat{\mathcal{D}}_- \hat{\Lambda}_P^* \hat{\mathcal{D}}_-^{-1} \hat{\mathbf{\Psi}} \\
&= -\hat{\Lambda}_P \hat{\mathbf{\Psi}}.
\end{aligned}$$

Remark:

P. Van Nieuwenhuizen (1984) defines a Majorana particle as a particle such that its Dirac adjoint is equal to its Majorana adjoint (see Appendix A for the definitions of adjoints). According to his definition, a Majorana pinor is such that $\psi^c = \pm\psi$, rather than the more commonly used condition $\psi^c = \psi$.

C. Existence of Majorana particles

Double beta decay occurs usually with the emission of two neutrinos. But if the neutrino associated with a beta decay is reabsorbed to produce a second beta decay, then no neutrino is emitted, and the process is called neutrinoless double beta decay. Diagrams of neutrinoless double beta decay in two different reactions are given in Fig. 6 and Fig. 7. If the same neutrino is emitted and absorbed, it has to be a particle identical with its antiparticle, i.e. it has to be a Majorana particle. The possible observation neutrinoless double beta decay is analyzed in a paper by Klapdor-Kleingrothaus (1994).

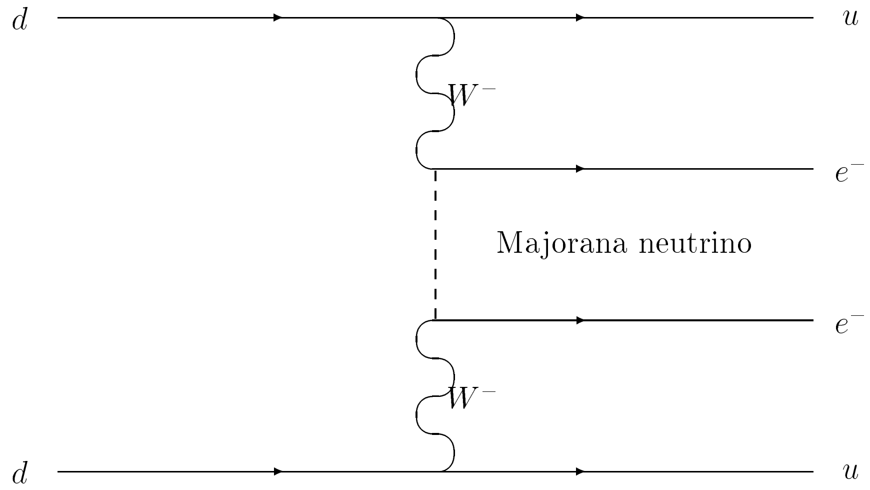

FIG. 6. Quark diagram for neutrinoless double beta decay ($2n \to 2p{+}2e^-$in ^{82}Se $\to$ ^{82}Kr+$2e^-$).

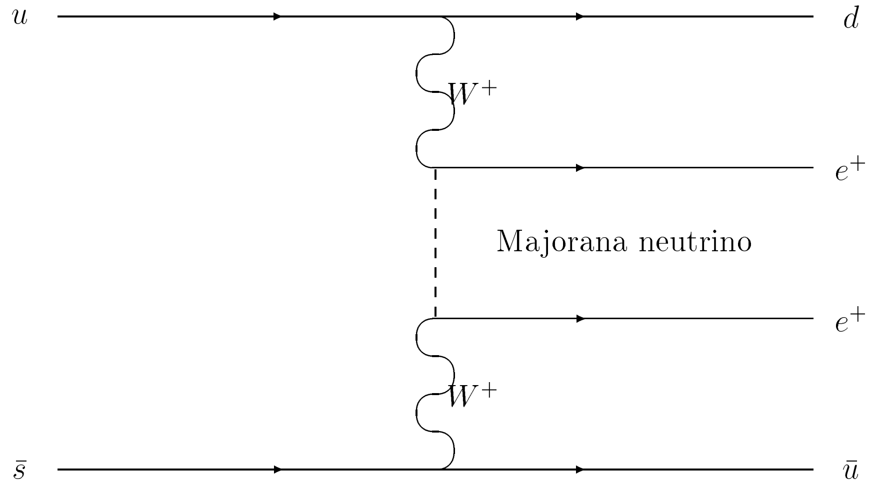

FIG. 7. We note also another type of neutrinoless double beta decay: Quark diagram for $K^+ \to \pi^- + 2e^+$.

V. DOUBLE COVERS OF THE LORENTZ GROUP

After having identified Pin (3,1) and Pin (1,3) as two inequivalent covers of the Lorentz group, we have to review briefly the other double covers of the Lorentz group.

We can look at Pin (s,t) and Pin (t,s) as extension of $O(s,t)$ (equivalently $O(t,s)$) by $\mathbf{Z}_2$ in the short exact sequence

$$1 \to \mathbf{Z}_2 \to G \to O(s,t) \to 1, \quad \mathbf{Z}_2 = \{1, -1\}$$

Two extensions G and G' are equivalent if and only if G and G' are isomorphic *and* the two sequences

$$\begin{array}{c} 1 \to \mathbf{Z}_2 \to G \to O(s,t) \to 1 \\ \downarrow \circ \wr \circ \downarrow \\ 1 \to \mathbf{Z}_2 \to G' \to O(s,t) \to 1 \end{array}$$

are made of two commutative diagrams: the two maps $\mathbf{Z}_2 \to G \to G'$ and $\mathbf{Z}_2 \to \mathbf{Z}_2 \to G'$ are identical (up to isomorphisms)—and the same property for the other diagram.

There are eight double covers of the Lorentz group called Pin^{abc} by L. Dabrowski (1988) and characterised by

$$\Lambda_P^2 = a, \quad \Lambda_T^2 = b, \quad (\Lambda_P \Lambda_T)^2 = c, \quad a, b, c \epsilon \mathbf{Z}_2.$$

In the fourth column of the following table, we give the names of the corresponding finite groups with elements $\pm 1, \pm\Lambda_P, \pm\Lambda_T$

Λ_P^2	Λ_T^2	$(\Lambda_P\Lambda_T)^2$	group	Remark
1	1	1	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$	$\Lambda_T\Lambda_P = \Lambda_P\Lambda_T$
1	-1	-1	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$	idem
-1	1	-1	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$	idem
-1	-1	1	$\mathbf{Z}_2 \oplus \mathbf{Z}_4$	idem
-1	-1	-1	quaternion	$\Lambda_T\Lambda_P = -\Lambda_P\Lambda_T$
-1	1	1	dihedral generating Pin (3,1)	idem
1	-1	1	dihedral generating Pin (1,3)	idem
1	1	-1	dihedral	idem

The following requirements identify the Pin groups, called cliffordian by Dabrowski (1988):

$$\Lambda_T\Lambda_P = -\Lambda_P\Lambda_T$$

$$\Lambda^2 \neq \Lambda_T^2$$

By relating $\Lambda_P, \Lambda_T \;\epsilon\; Pin(s,t)$ to $P, T \;\epsilon\; O(s\,t)$ respectively (or $\hat{\Lambda}_P, \hat{\Lambda}_T$ to P, T) we give the explicit Pin structure (group, homomorphism $\mathcal{H}$: Pin group $\rightarrow$ Orthonormal group). There is an extensive literature (see Appendix D) on Pin structures, not only for Pin groups covering Lorentz groups, but also for Pin groups covering $O(m,n)$ with arbitrary values of m and n.

VI. CONCLUSION

Our search for *experiments* distinguishing Ψ and $\hat{\Psi}$-particles has so far given only a slim result: The observation of neutrinoless double beta decay would identify the electron neutrino exchanged in the process as a Pin (3,1) particle.

The sought-for experiments may come from far afield. It could come from systems with topologically nontrivial configuration space: It has been shown (DeWitt-Morette and DeWitt, 1990) that the vacuum expectation values of Fermi currents on Klein bottles $\times\mathbf{R}^2$ computed with Pin (3,1) and Pin (1,3) are totally different. With Dirac fields acted upon by Pin (3,1) the only nonvanishing current is a pseudo scalar current. In the other case the only nonvanishing current is tensor valued. Moreover the current intensities as functions of position are also totally different. Laboratory systems with nontrivial configuration spaces exist but their investigations require expertise in condensed matter physics, atomic and molecular physics and are topics for major research projects.

In addition to offering a classification of particles finer than the Spin group, the Pin groups provide simple and straightforward definitions of time reversal, space reversal, charge conjugation and adjoint pinors. The title of this paper could be "Spin or Pin?, and a C, P, T tutorial".

VII. ACKNOWLEDGEMENTS

It is a pleasure to thank our correspondents for their interest in the subject, but the list of their names would necessarily be incomplete, and serve little purpose other than name dropping. Special thanks are due to Steven Carlip whose suggestions have "vastly improved" the first draft of this paper.

APPENDIX A: DIRAC AND MAJORANA ADJOINTS. SEVERAL DEFINITIONS OF CHARGE CONJUGATION.

The adjoint $\bar{\Psi}$ of a contravariant pinor field Ψ is a covariant pinor field (copinor) such that the duality pairing $< \bar{\Psi}, \Psi >$ is a scalar:

$$\langle \bar{\Psi}, \Psi \rangle := \int_{\text{spacetime}} \sum_A \bar{\Psi}_A(x) \Psi^A(x) \mathrm{dx} \tag{A1}$$

where

$\Psi(x) := \sum_A \Psi^A(x) e_A$;

the adjoint $\bar{\Psi}, \Psi$ is defined so that $< \bar{\Psi}, \Psi >$ is invariant under Lorentz transformations, i.e.

$$\langle \bar{\Psi}, \Psi \rangle = \langle \overline{\Lambda\Psi}, \Lambda\Psi \rangle. \tag{A2}$$

When $\Lambda \in Pin^{\uparrow}(3,1)$ (i.e. the double cover of the orthochronous Lorentz group), the solutions of the equation

$$\overline{\Lambda\Psi} = \bar{\Psi}\Lambda^{-1} \tag{A3}$$

most frequently used are the Dirac adjoint $\bar{\Psi}_D$, and the Majorana adjoint $\bar{\Psi}_M$:

$$\bar{\Psi}_D := \Psi^{\dagger} \mathcal{H}_{-} \tag{A4}$$

where $\dagger$ stands for complex conjugate transposed and $\mathcal{H}_{-}$ is defined by

$$\mathcal{H}_{\pm} \Gamma_{\alpha} \mathcal{H}_{\pm}^{-1} = \pm \Gamma_{\alpha}^{\dagger}, \tag{A5}$$

$$\bar{\Psi}_M := \Psi^T \mathcal{T}_{-} \tag{A6}$$

where T stands for transposed and $\mathcal{T}$ is defined by

$$\mathcal{T}_{\pm} \Gamma_{\alpha} \mathcal{T}_{\pm}^{-1} = \pm \Gamma_{\alpha}^{T}. \tag{A7}$$

See Table 1, for explicit expressions of $\mathcal{H}_{-}$ and $\mathcal{T}_{-}$.

When Λ is not restricted to be an element of $\mathrm{Pin}^+(3,1)$, (see Choquet-Bruhat and DeWitt-Morette (1989), pp. 36-37)

$$\bar{\Psi}_D = a_\lambda \Psi^+ \mathcal{H}_-.$$

where a_λ is a representation of Pin (3,1) in $\mathbf{Z}_2 = \{1, -1\}$; namely $a_\lambda = 1$ for $\lambda \epsilon$ $\mathrm{Pin}^+(3.1)$ and $a_\lambda = -1$ otherwise.

It is possible to define the Dirac adjoint of a pinor field on a hyperbolic manifold which admits a time orientation. A pin frame is future [resp. past] oriented if it projects onto a future [resp. past] oriented Lorentz frame. To each pin frame is associated a number equal to 1 if it is future oriented and to a -1 otherwise. The Dirac adjoint is affected by this sign. For the explicit expression, when pinors are defined as equivalence classes, see Choquet-Bruhat and DeWitt-Morette (1989).

Similar conclusions hold for Pin (1,3) and for Majorana's adjoints.

Charge conjugation is often defined in terms of the Dirac adjoint

$$\Psi^c = \mathcal{T}_-^{-1} \bar{\Psi}_D^T, \tag{A8}$$

and occasionally in terms of the Majorana adjoint

$$\Psi^c = \mathcal{H}_-^{-1} \bar{\Psi}_M^\dagger. \tag{A9}$$

We shall prove that both definitions are equivalent to the natural definition (4.3), namely $\Psi^c = \mathcal{D}_+ \Psi^*$.

It follows from the definitions (A5) and (A7) that

$$-\left(\mathcal{H}_-^T\right)^{-1} \Gamma_\alpha^T \mathcal{H}_-^T = \left(\mathcal{H}_-^T\right)^{-1} \left(\mathcal{T}_- \Gamma_\alpha \mathcal{T}_-^{-1}\right) \mathcal{H}_-^T = \Gamma_\alpha^{\dagger T} = \Gamma_\alpha^*,$$

whence by definition (IV.3)

$$\mathcal{D}_+ = \mathcal{T}_-^{-1} \mathcal{H}_-^T.$$

Hence

$$\mathcal{T}_-^{-1}\bar{\Psi}_D^T = \mathcal{T}_-^{-1}(\Psi^\dagger\mathcal{H}_-)^T = \mathcal{T}_-^{-1}\mathcal{H}_-^T\Psi^* = \mathcal{D}_+\Psi^*.$$

Similarly, it follows from the definitions (A7) and (A5) that

$$-\left(\mathcal{T}_-^\dagger\right)^{-1}\Gamma_\alpha^\dagger\mathcal{T}_-^\dagger = \left(\mathcal{T}_-^\dagger\right)^{-1}\left(\mathcal{H}_-\Gamma_\alpha\mathcal{H}_-^{-1}\right)\mathcal{T}_-^\dagger = \Gamma_\alpha^{T\dagger} = \Gamma_\alpha^*,$$

whence by definition (IV.3)

$$\mathcal{D}_+ = \mathcal{H}_-^{-1}\mathcal{T}_-^\dagger.$$

Hence

$$\mathcal{H}_-^{-1}\bar{\Psi}_M^\dagger = \mathcal{H}_-^{-1}(\Psi^T\mathcal{T}_-)^\dagger = \mathcal{H}_-^{-1}\mathcal{T}_-^\dagger\Psi^* = \mathcal{D}_+\Psi^*.$$

In $Pin(1,3)$,

$$\bar{\hat{\Psi}}_D := \hat{\Psi}^\dagger\hat{\mathcal{H}}_+$$

and

$$\bar{\hat{\Psi}}_M := \hat{\Psi}^T\hat{\mathcal{T}}_-$$

so that $\bar{\hat{\Psi}}_D$ and $\bar{\hat{\Psi}}_M$ satisfy equations which relate to the Dirac equation for $\hat{\Psi}$ in the same way that the equations satisfied by $\bar{\Psi}_D$ and $\bar{\Psi}_M$ relate to the Dirac equation for Ψ (see Table 1). Consequently

$$\langle\bar{\hat{\Psi}}_D, \hat{\Psi}\rangle$$

is invariant under orthochronous Lorentz transformations (Lorentz transformations connected to $\mathbb{1}$ or P) but

$$\langle\bar{\hat{\Psi}}_M, \hat{\Psi}\rangle$$

is invariant under Lorentz transformations connected to $\mathbb{1}$ and T. See for instance [Choquet-Bruhat and DeWitt-Morette, pp. 36–37].

Remark

The attribute "Majorana" is applied to representations, particles, and adjoint pinors with different meanings. For instance, if one calls "Majorana" a real representation of the gamma matrices, then Majorana pinors such that $\Psi^c = \Psi$ do not necessarily define a real representation.

Remark

In a change of representation defined by

$$\Psi \longrightarrow \Lambda\Psi \quad , \quad \Gamma_\alpha \longrightarrow \Lambda\Gamma_\alpha\Lambda^{-1}$$

then

$$\begin{aligned}
\mathcal{D}_\pm &\longrightarrow \Lambda\mathcal{D}_\pm \left(\Lambda^*\right)^{-1} \\
\mathcal{T}_\pm &\longrightarrow \left(\Lambda^T\right)^{-1} \mathcal{T}_\pm\Lambda^{-1} \\
\mathcal{H}_\pm &\longrightarrow \left(\Lambda^\dagger\right)^{-1} \mathcal{H}_\pm\Lambda^{-1}.
\end{aligned}$$

APPENDIX B: "CPT" TRANSFORMATION

The "CPT" theorem is in the terminology of this paper a CPΘ theorem where Θ is the antiunitary time reversal (see paragraph 2.1). There are different equivalent, formulations of the "CPT" theorem. For example H.P. Stapp (1962) stated: "The "CPT" theorem states that for every process occuring in nature there is an allowed dual process in which the particles of the first are replaced by their respective antiparticles, all spins are reversed, and paths are changed to their images under inversion through the origin of space-time." This theorem was proved first by G. Lüders (1954) and W. Pauli (1955), it was discussed by Lüders and Zumino (1957) and proved rigorously in axiomatic field theory by R. Jost (1957) and by F.J. Dyson (1958). In 1961, Stapp showed that the CPT theorem is a consequence of postulates of the S-matrix much weaker than those of field theory; namely his proof depends on unitarity, macroscopic relativistic invariance, and a very weak analyticity requirement on the mass-shell scattering functions. Here we show that a CPΘ transformation (the transformation usually called CPT) is simply an orientation preserving Lorentz transformation. Indeed a CPΘ transformation of a spinor field Ψ is given by

$$\begin{aligned}\mathcal{D}_+(\Lambda_P(\Theta\Psi^*))^* &= (\mathcal{D}_+\Lambda_P^*\Lambda_T^*\mathcal{D}_+^*)\Psi \\ &= (\mathcal{D}_+\Lambda_P^*\Lambda_T^*\mathcal{D}_+^{-1})\mathcal{D}_+\mathcal{D}_+^*\Psi \\ &= \Lambda_P\Lambda_T\Psi.\end{aligned}$$

On a spinor field $\hat{\Psi}$ one has

$$\begin{aligned}\hat{\mathcal{D}}(\hat{\Lambda}_P(\hat{\Theta}\hat{\Psi}^*))^* &= (\hat{\mathcal{D}}_-\hat{\Lambda}_P^*\hat{\Lambda}_T^*\hat{\mathcal{D}}_-^*)\hat{\Psi} \\ &= (\hat{\mathcal{D}}_-\hat{\Lambda}_P^*\hat{\Lambda}_T^*\hat{\mathcal{D}}_-^{-1})\hat{\mathcal{D}}_-\hat{\mathcal{D}}_-^*\hat{\Psi} \\ &= \hat{\Lambda}_P\hat{\Lambda}_T\hat{\Psi}.\end{aligned}$$

The fact that CPΘ (i.e. the usual CPT) is an orientation preserving Lorentz transformation implies that it is unitary. It also implies that the existence of antiparticles is a consequence of orientation preserving Lorentz transformations.

APPENDIX C: PIN(3,1) AND PIN(1,3).

Table 1 summarizes the results established in the previous paragraphs for Pin(3,1) and lists the corresponding results for Pin(1,3). Generalizations to $Pin(s,t)$ and $Pin(t,s)$ for arbitrary s and t can be obtained by combining the results of this paper with an earlier paper by DeWitt-Morette and Gwo (1990).

TABLES

	Pin(3,1)	Pin(1,3)
	$\Lambda_P^2 = -\mathbb{1}, \Lambda_T^2 = -\mathbb{1},$ $\Lambda_{PT}^2 = -\mathbb{1}, \Theta^2 = -\mathbb{1}$	$\hat{\Lambda}_P^2 = \mathbb{1}, \hat{\Lambda}_T^2 = \mathbb{1},$ $\hat{\Lambda}_{PT}^2 = -\mathbb{1}, \hat{\Theta}^2 = -\mathbb{1}$
Dirac Equation	$[\Gamma^\alpha(\partial_\alpha + iqA_\alpha) - m]\Psi = 0$	$[\hat{\Gamma}^\alpha(\partial_\alpha + iqA_\alpha) + im]\hat{\Psi} = 0$
	$\mathcal{D}_+^{-1}\Gamma_\alpha \mathcal{D}_+ = \Gamma_\alpha^*$ $\mathcal{T}_-\Gamma_\alpha \mathcal{T}_-^{-1} = -\Gamma_\alpha^T$ $\mathcal{H}_-\Gamma_\alpha \mathcal{H}_-^{-1} = -\Gamma_\alpha^\dagger = -\Gamma_\alpha^{*T}$	$\hat{\mathcal{D}}_-^{-1}\hat{\Gamma}_\alpha \hat{\mathcal{D}}_- = -\hat{\Gamma}_\alpha^*$ $\hat{\mathcal{T}}_-\hat{\Gamma}_\alpha \hat{\mathcal{T}}_-^{-1} = -\hat{\Gamma}_\alpha^T$ $\hat{\mathcal{H}}_+\hat{\Gamma}_\alpha \hat{\mathcal{H}}_+^{-1} = \hat{\Gamma}_\alpha^\dagger = \hat{\Gamma}_\alpha^{*T}$
In the Dirac Representation	$\mathcal{D}_+ = \Gamma_2$ $\mathcal{T}_- = i\Gamma_2\Gamma_4$ $\mathcal{H}_- = i\Gamma_4$	$\hat{\mathcal{D}}_- = i\hat{\Gamma}_2$ $\hat{\mathcal{T}}_- = i\hat{\Gamma}_2\hat{\Gamma}_4$ $\hat{\mathcal{H}}_+ = \hat{\Gamma}_4$
Dirac Adjoint	$\bar{\Psi}_D = \Psi^\dagger \mathcal{H}_-$	$\bar{\hat{\Psi}}_D = \hat{\Psi}^\dagger \hat{\mathcal{H}}_+$
Majorana Adjoint	$\bar{\Psi}_M = \Psi^T \mathcal{T}_-$	$\bar{\hat{\Psi}}_M = \hat{\Psi}^T \hat{\mathcal{T}}_-$
Equations Satisfied by $\bar{\Psi}_D, \bar{\Psi}_M,$ $\bar{\Psi}_D^c$ and $\bar{\Psi}_M^c$	$\bar{\Psi}_D[\Gamma^\alpha(\overleftarrow{\partial}_\alpha - iqA_\alpha) + m] = 0$ $\bar{\Psi}_M[\Gamma^\alpha(\overleftarrow{\partial}_\alpha + iqA_\alpha) + m] = 0$ $\bar{\Psi}_D^c[\Gamma^\alpha(\overleftarrow{\partial}_\alpha + iqA_\alpha) + m] = 0$ $\bar{\Psi}_M^c[\Gamma^\alpha(\overleftarrow{\partial}_\alpha - iqA_\alpha) + m] = 0$	$\bar{\hat{\Psi}}_D[\hat{\Gamma}^\alpha(\overleftarrow{\partial}_\alpha - iqA_\alpha) - im] = 0$ $\bar{\hat{\Psi}}_M[\hat{\Gamma}^\alpha(\overleftarrow{\partial}_\alpha + iqA_\alpha) - im] = 0$ $\bar{\hat{\Psi}}_D^c[\hat{\Gamma}^\alpha(\overleftarrow{\partial}_\alpha + iqA_\alpha) - im] = 0$ $\bar{\hat{\Psi}}_M^c[\hat{\Gamma}^\alpha(\overleftarrow{\partial}_\alpha - iqA_\alpha) - im] = 0$
Charge Conjugate	$[\Gamma^\alpha(\partial_\alpha - iqA_\alpha) - m]\Psi^c = 0$ $\Psi^c = \mathcal{D}_+\Psi^*$ $= \mathcal{T}_-^{-1}\bar{\Psi}_D^T = \mathcal{H}_-^{-1}\bar{\Psi}_M^\dagger$	$[\hat{\Gamma}^\alpha(\partial_\alpha - iqA_\alpha) + im]\hat{\Psi}^c = 0$ $\hat{\Psi}^c = \hat{\mathcal{D}}_-\hat{\Psi}^*$ $= \hat{\mathcal{T}}_-^{-1}\bar{\hat{\Psi}}_D^T = \hat{\mathcal{H}}_+^{-1}\bar{\hat{\Psi}}_M^\dagger$
Time Reversal	$\Psi'(Tx) = \Theta\Psi^*(x)$ $\Theta = \Lambda_T \mathcal{D}_+$ $\Lambda_T = \Gamma_1\Gamma_2\Gamma_3$	$\hat{\Psi}'(Tx) = \hat{\Theta}\hat{\Psi}^*(x)$ $\hat{\Theta} = \hat{\Lambda}_T \hat{\mathcal{D}}_-$ $\hat{\Lambda}_T = \hat{\Gamma}_1\hat{\Gamma}_2\hat{\Gamma}_3$
In the Dirac Representation	$\Theta = \Gamma_1\Gamma_3$	$\hat{\Theta} = i\hat{\Gamma}_1\hat{\Gamma}_3$

TABLE I. Pin(3,1) and Pin(1,3)

The difference between $Pin(s,t)$ and $Pin(t,s)$ is due to the fact that the mapping

$$\Gamma_\alpha \longrightarrow \hat{\Gamma}_\alpha = i\Gamma_\alpha$$

is a bijection but not an isomorphism, a fact that is strikingly apparent in Table 1: one does not obtain the $Pin(t,s)$ equations by putting hats on the $Pin(s,t)$ equations.

$\{\Gamma_\alpha\}$ is said to be a unitary representation of $\{\gamma_\alpha\}$ if

$$\Gamma_\alpha \Gamma_\alpha^\dagger = \mathbb{1}.$$

For the free particle Hamiltonian of a spin-1/2 particle to be self-adjoint, one must have $\Gamma^{\mu\dagger} = \Gamma^0\Gamma^\mu\Gamma^0$ (likewise $\hat{\Gamma}^{\mu\dagger} = \hat{\Gamma}^0\hat{\Gamma}^\mu\hat{\Gamma}^0$); in flat spacetime, with $g_{\mu\nu} = \eta_{\mu\nu}$ (and $\hat{g}_{\mu\nu} = \hat{\eta}_{\mu\nu}$), this implies ${\Gamma_\mu}^\dagger = \Gamma_0\Gamma_\mu\Gamma_0$ (likewise ${\hat{\Gamma}_\mu}^\dagger = \hat{\Gamma}_0\hat{\Gamma}_\mu\hat{\Gamma}_0$), hence ${\Gamma_\mu}^\dagger = (\Gamma_\mu)^{-1}$ (and ${\hat{\Gamma}_\mu}^\dagger = (\hat{\Gamma}_\mu)^{-1}$). In any unitary representation of $Pin(3,1)$, Γ_i $(i = 1,2,3)$, is hermitian, and Γ_4 is antihermitian. See Table 1 for matrix representations of $\mathcal{D}_+, \hat{\mathcal{D}}_-, \mathcal{T}_-, \hat{\mathcal{T}}_-, \mathcal{H}_-$ and $\hat{\mathcal{H}}_+$, valid in any unitary representation unless otherwise indicated. We list only the ones needed for constructing solutions of the Dirac charge conjugate equation and adjoint equation, given a solution of the Dirac equation.

APPENDIX D: SOME RELATED LITERATURE, OTHER THAN QUOTED IN THE REFERENCES.

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